

# 1 R/C Section 7.3 # 2 Do pg 265 1 - 8 ALSO write a complete explanation of telescoping sums. Use pblm 8 in your demo. answers to pblm 8 are: $2^{100} - 1, \frac{-99}{100}, \frac{-100}{101} \quad 1/27$	# 3 Pg 265 9, 11 - 16 hint on pblm 9: $x_n = 4$ and $x_0 = 0$ # 4 R/C Pg 250 - 253 then R/C pg 255 - 260 4 problems are stated Collect #1,2 before discussion. Talk about pblm 8 1/28	# 5 Pg 253 - 254 1, 2, 6, 7 ----- # 6 Pg 276 - 277 1 - 14 Pblm 14 is important. Do a complete job on it! ----- RiemannC program Lecture Day #1 1/29	# 7 Rewrite "Summary" IN YOUR OWN WORDS. Pg 278 - 279 ----- # 8 GQ Pg 279-280 1, 2, 3 RE Pg 280-281 1 - 6 Go thru #6 pblm by pblm in class to force student focus on being prepared. 1/30	RE Pg 281 15 # 9 ----- ALSO do "matching" on the side chalk board # 10 Collect hw at beginning of period. Grade hard - emphasis on "doing your own work" 1/31
# 11 Pg 287 - 288 1 - 19 First Fund Thm of Calc for pblm 14: Make Reimann Sums using 3 sec tions. Show intermediate values. Use FnInt(to compare area values. For pblms 15, 16, 17: <i>also</i> evaluate using FnInt(	# 12 Pg 291 - 292 10 - 18 Second Fund Thm of Calc 2/4	R/C Pg 292-297 # 13 ----- # 14 Pg 297 1 - 12, 14 - 17 Important Lecture Have pg 291 - 292 pblms 10 - 18 put on boards Proof of FTC 2/5	R/C Pg 298 - 300 # 15 ----- Pg 300 - 301 1 - 30 , 32 # 16 Remember to include the "+ c " indefinite integral 2/6	# 17 , 18, 19 "Honor Quiz" 2/7
Holiday # 20 "Calculator 101" 2/10	MAA test # 21 GQ C8 Pg 302 1, 3, 4, 6, 7, 9 GQ Ch 1-8 Pg 302-303 1, 2, 3 2/11	#22 "Semi Circle the Wagons" 2/12	#23 RE Ch 1-8 Pg 304-306 1 - 3, 5 - 7, 9, 12 - 14, # 24 Pg 304-306 32-36, 38-40, 43, 50, 52 - 56 Upper limit on # 40 should be $\sqrt{\pi}$. 2/13	# 25, #26 "Translating ..." # 27 Ff f' f''(un)ctions 2/14
Holiday 2/17	Put last year's exam on the boards. 2/18	Exam # 1 Quarter 3 2003 2/19	Advanced Placement Calculus AB Chapters 7 & 8 First Exam Third Quarter 2/20 2/21	

RiemannC

Program for TI 82

```

1→L
2→R
FnOff
Prompt A
Prompt B
Lbl 1
Prompt N
(B-A)/N→H
Disp "L(1) OR R(2)"
Input W
0→S
ClrDraw
DrawF Y1
If W≠L
Goto 2
N-1→T
0→U
H→Z
bl 3
A+UH→X
Y1→Q
S+QH→S
X→P
Line(P,0,P,Q)
Line(P,Q,P+Z,Q)
Line(P+Z,Q,P+Z,0)
IS>(U,T)
Goto 3
Pause
Disp "SUM"
Disp S
Pause
Goto 1
Lbl 2
If W≠R
Goto 1
N→T
1→U
-H→Z
Goto 3

```

TI 85 Program

```

1→L
2→R
FnOff
Prompt A
Prompt B
Lbl ONE
Prompt N
(B-A)/N→H
Disp "L(1) OR R(2)"
Input W
0→S
ClDrw
DrawF y1
If W≠L
Goto TWO
N-1→T
0→U
H→Z
Lbl THR
A+U*H→X
y1→Q
S+Q*H→S
X→P
Line(P,0,P,Q)
Line(P,Q,P+Z,Q)
Line(P+Z,Q,P+Z,0)
IS>(U,T)
Goto THR
Pause
Disp "SUM"
Disp S
Pause
Goto ONE
Lbl TWO
If W≠R
Goto ONE
N→T
1→U
-H→Z
Goto THR

```

1. Key in the program for your machine.

This is zero → 0

IS>(is a program control word found on the 82 under "Program CTL"

The 82 only allows single letter variable names so VH means V times H.

On the 85 you must explicitly state the multiply.

2. Set $Y_1 = x^2$ (on the 85 use CAP X for this definition.)

A = 0

B = 3

3. Set the window: 0,3,1,0,9,3

4. Copy the following table to your paper:

Put this value on the window

Try #	N	L(1)R(2)	Result
1	6	1	6.875
2	6	2	11.375
3	12	1	7.906
4	12	2	10.156
5	18	1	8.26
6	18	2	9.76
7	100	1	8.865
8	100	2	9.135
9	1000	1	8.9865
			9.0135

1000. Put this value on the window

5. Run the program RiemannC and record your results in your table.

50 students will know they got the program in their calculator properly.

Biggs
File Name: RIEMANNC.89P

```
( )  
Prgm  
1→l  
2→r  
FnOff  
Prompt a  
Prompt b  
Lbl o  
Prompt n  
(b-a)/n→h  
Disp "l(1) or r(2)"  
Input w  
0→s  
ClrDraw  
DrawFunc y1(x)  
If w≠1  
Goto d  
n-1→t  
0→v  
h→z  
Lbl k  
a+v*h→x  
y1(x)→q  
s+q*h→s  
x→p  
Line p,0,p,q  
Line p,q,p+z,q  
Line p+z,q,p+z,0
```

If v<t Then

```
v+1→v  
Goto k  
Else  
Pause  
Disp "sum"  
Disp s  
EndIf  
Pause  
Goto o  
Lbl d  
If w≠r  
Goto o  
n→t  
1→v  
-h→z  
Goto k  
EndPrgm
```

The following 4 pages are based on a problem found in the Advanced Placement Calculus "Acorn Book" page 38 of the 2000-2001 acorn book and on page 41 of the 2002-2003 acorn book.

I plan to use the material in this fashion.....

I will photo the next page which is where the functions are defined and the questions are asked. I will hand that page out to the class. I will give them 20 minutes to work independently on problem sections I and II only.

After the 20 minute period, we will discuss what they came up with and compare to my version. I will take this opportunity to re-establish why each of the items is as it is.

The homework for the evening will be to complete the worksheet.

The 3 pages following the "handout" is a list of my answers and brief comments explaining my method.

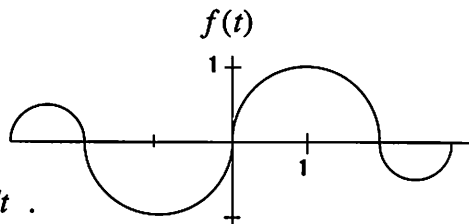
If you use this, I would like to hear from you concerning the success you had and perhaps improvements I could make in the exercise. My email address is teeitup98@telocity.com.

Thank you for your consideration.

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The graph of the function f on the right consists of four semicircles.

Given: $g(x) = \int_0^x f(t) dt$, $h(x) = \int_1^x f(t) dt$ and $j(x) = \int_0^{2x} f(t) dt$.



- I
 - a) Consider a circle of radius one-half. What is the area of its semicircle?
 - b) Consider a circle of radius one. What is the area of its semicircle?
 - c) Is the function $f(t)$ odd, even, or neither?

- II
 - a) How does the answer to part (Ic) effect $g(x)$?
 - b) Find: $g(0)$, $g(2)$, $g(3)$, $g(-2)$, $g(-3)$
 - c) Make a number line for $g'(x)$. Mark zeros as well as indicate positive and negative intervals.
 - d) Specifically state the intervals when $g(x)$ is increasing; decreasing.
 - e) Determine the *coordinates* of all extrema of $g(x)$. Justify, of course.
 - f) Determine the *coordinates* of all Points of Inflection of $g(x)$.
 - g) Sketch $g(x)$.
 - h) For what values of x is $g(x)$ non-negative.

- III
 - a) What is the domain of $h(x)$?
 - b) Compute: $h(1), h(0), h(2), h(3), h(-2), h(-3), h(-1)$
 - c) Draw a $h'(x)$ number line. Indicate $+$, $-$ as appropriate.
 - d) Compute *coordinates* of extrema. Justify, of course.
 - e) Draw a $h''(x)$ number line. Indicate $+$, $-$ as appropriate.
 - f) Compute *coordinates* of PI.
 - g) Sketch $h(x)$.

- IV
 - a) What is the domain of $j(x)$?
 - b) As in section III, compute the significant values of $j(x)$.
 - c) Draw $j'(x)$ number line, Indicate $+$, $-$ as appropriate.
 - d) Compute *coordinates* of extrema. Justify, of course.
 - e) Draw $j''(x)$ number line, Indicate $+$, $-$ as appropriate.
 - f) Compute *coordinates* of PI.
 - g) Sketch $j(x)$.

* Ask your parents about Ward Bond's famous words.

I a) $\frac{\pi}{8}$

I b) $\frac{\pi}{2}$

I c) f is an odd function because the function reflects through the origin.
For example, $(1, 1)$ reflects to $(-1, -1)$.

II a) g will be an even function so $g(a) = g(-a)$.

II b) $g(0) = 0$ because $g(0) = \int_0^0 f(t) dt$.

$$g(2) = \int_0^2 f(t) dt \text{ which is } \frac{\pi}{2} \text{ (from part b) so } g(-2) = g(2) = \int_0^2 f(t) dt = \frac{\pi}{2}$$

$$g(3) = \int_0^3 f(t) dt = \frac{\pi}{2} - \frac{\pi}{8} = \frac{3\pi}{8} \text{ subtracting the smaller semicircle from the larger so}$$

$$g(-3) = g(3) = \int_0^3 f(t) dt = \frac{\pi}{2} - \frac{\pi}{8} = \frac{3\pi}{8}.$$

II c) $\frac{+ \ 0 \ - \ 0 \ + \ 0 \ - \ 1}{-3 \ -2 \ \hat{0} \ \ 2 \ \ 3}$

II d) $g'(x) > 0$ for $-3 < x < -2$ and for $0 < x < 2$ thus g is increasing in these intervals.
 $g'(x) < 0$ for $-2 < x < 0$ and for $2 < x < 3$ thus g is decreasing in these intervals.

II e) $g(-3) = g(3) = \frac{3\pi}{8}$. The left end point is a minimum because g increases from it and the right end point

is a minimum because g decreases to it. coordinates: $\left(3, \frac{3\pi}{8}\right)$ and $\left(-3, \frac{3\pi}{8}\right)$

$g(-2) = g(2) = \frac{\pi}{2}$ are maximums by $f'(x)$ test. coordinates: $\left(2, \frac{\pi}{2}\right)$ and $\left(-2, \frac{\pi}{2}\right)$

$g(0) = 0$ by $f'(x)$. coordinates $(0, 0)$

II f) PI occur in g when $g''(x)$ changes sign. This is when $f'(x)$ changes sign. From the our graph, this is when f changes from increasing to decreasing (or vice versa).

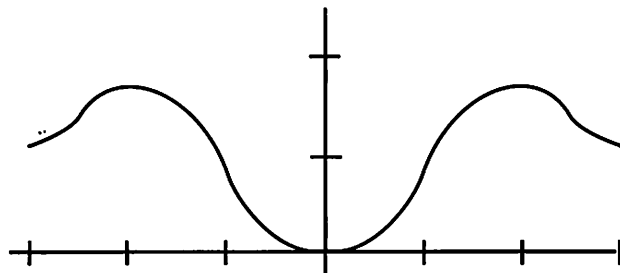
PI occur when x is an element of the set: $\{-2.5, -1, 1, 2.5\}$. Note: $\frac{\pi}{2} \cdot \frac{1}{2} = \frac{\pi}{4}$ and $\frac{7\pi}{16} = \frac{\pi}{2} - \frac{\pi}{16}$

Coordinates: $\left(-2.5, \frac{7\pi}{16}\right), \left(-1, \frac{\pi}{4}\right), \left(1, \frac{\pi}{4}\right), \left(2.5, \frac{7\pi}{16}\right)$

$\frac{\text{up} \ \text{dn} \ \text{up} \ \text{dn} \ \text{up}}{+ \ 0 \ - \ 0 \ + \ 0 \ - \ 0 \ +}$
 $\frac{\text{up} \ \text{dn} \ \text{up} \ \text{dn} \ \text{up}}{-3 \ -2.5 \ -2 \ -1 \ \hat{0} \ 1 \ 2 \ 2.5 \ 3}$

Note: $g''(x)$ DNE for $x \in \{-2, 0, 2\}$ because the tangent line to the graph of f becomes vertical.

II g)



II h) g is non-negative for all $x \in [-3, 3]$.

III a) $D_h x \in [-3, 3]$ There is no change because upper bound is x (rather than $x+1$, for example)

III b) $h(1) = \int_1^1 f(t) dt = 0 \quad (1, 0) \quad h(0) = \int_1^0 f(t) dt = -\int_0^1 f(t) dt = -\frac{\pi}{4} \quad \left(0, -\frac{\pi}{4}\right)$

$h(2) = \int_1^2 f(t) dt = \frac{\pi}{4} \quad \left(2, \frac{\pi}{4}\right) \quad h(3) = \int_1^3 f(t) dt = \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8} \quad \left(3, \frac{\pi}{8}\right)$

$h(-1) = \frac{\pi}{4} - \frac{\pi}{4} = 0 \quad (-1, 0) \quad h(-2) = -\left(-\frac{\pi}{4}\right) \quad \left(-2, \frac{\pi}{4}\right)$

$h(-3) = \text{this is } h(-2) - \frac{\pi}{4} = \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8} \quad \left(-3, \frac{\pi}{8}\right)$

III c) This is the same as it was for II c because $h'(x) = f(x)$ just as $g'(x) = f(x)$.

$$\begin{array}{ccccccc} 0 & + & 0 & - & 0 & + & 0 & - & 0 \\ -3 & -2 & & 0 & & 2 & 3 & & \end{array}$$

III d) $\left(-3, \frac{\pi}{8}\right)$ min. end pt increasing from it. $\left(-2, \frac{\pi}{4}\right)$ max $f'(x)$ test

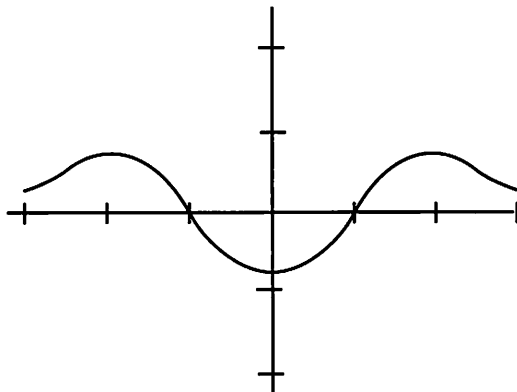
$\left(0, -\frac{\pi}{4}\right)$ min $f'(x)$ test $\left(2, \frac{\pi}{4}\right)$ max $f'(x)$ test $\left(3, \frac{\pi}{8}\right)$ min end pt decrease to it.

III e) $h''(x)$ $\begin{array}{cccccccc} \text{up} & \text{dn} & & \text{up} & \text{dn} & \text{up} & & \\ \text{+} & \text{0} & \text{-} & \text{0} & \text{+} & \text{0} & \text{-} & \text{0} & \text{+} \\ -3 & -2.5 & -2 & -1 & 0 & 1 & 2 & 2.5 & 3 \end{array}$ Note: This is exactly the same as $g''(x)$ (II f)

III f) As in part II f, the x values of the PI are the same.

$-\frac{\pi}{4} + \frac{\pi}{2} - \frac{\pi}{16} = \frac{3\pi}{16} \quad \left(-\frac{5}{2}, \frac{3\pi}{16}\right) \quad \frac{\pi}{4} - \frac{\pi}{16} = \frac{3\pi}{16} \quad \left(\frac{5}{2}, \frac{3\pi}{16}\right) \quad (-1, 0), (1, 0)$

III g) $h(x)$:



IV a) $-3 \leq t \leq 3$ so
 $-3 \leq 2x \leq 3$ $D_j x \in \left[-\frac{3}{2}, \frac{3}{2}\right]$
 $-\frac{3}{2} \leq x \leq \frac{3}{2}$

IV b) The $2x$ just changes the period of the function much in the same manner as $\sin x$ compared to $\sin 2x$.

$$j(0) = 0, j(-1) = j(1) = \int_0^{2 \cdot 1 = 2} f(t) dt = \frac{\pi}{2}, j\left(-\frac{3}{2}\right) = j\left(\frac{3}{2}\right) = \int_0^{2 \cdot \frac{3}{2} = 3} f(t) dt = \frac{3\pi}{8}$$

$$j\left(-\frac{1}{2}\right) = j\left(\frac{1}{2}\right) = \int_0^{2 \cdot \frac{1}{2} = 1} f(t) dt = \frac{\pi}{4} \quad j\left(-\frac{5}{4}\right) = j\left(\frac{5}{4}\right) = \int_0^{2 \cdot \frac{5}{4} = \frac{5}{2}} f(t) dt = \frac{\pi}{2} - \frac{\pi}{16} = \frac{7\pi}{16}$$

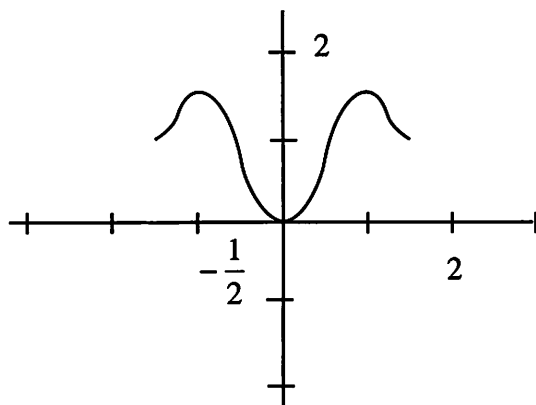
IV c) $j'(x)$ $\frac{[+ \ 0 \ - \ 0 \ + \ 0 \ -]}{-\frac{3}{2} \quad -1 \quad \underbrace{0}_{\frac{1}{4}} \quad 1 \quad \frac{3}{2}}$ $j'\left(\frac{1}{4}\right)$ would be the slope of $f(t)$ at $x = \frac{1}{2}$.
 From the original picture of f , f is clearly increasing. The $+$, $-$ signs alternate from there.

IV d) $\left(-\frac{3}{2}, \frac{3\pi}{8}\right)$ is min. end pt, increasing from it $\left(-1, \frac{\pi}{2}\right)$ is max. f' test. $(0, 0)$ min f' test.
 $\left(1, \frac{\pi}{2}\right)$ max, f' test. $\left(\frac{3}{2}, \frac{3\pi}{8}\right)$ is min. end pt. decreasing to it.

IV e) $\frac{\text{up} \quad \text{dn} \quad \text{up} \quad \text{dn} \quad \text{up}}{[+ \ 0 \ - \ 0 \ + \ 0 \ -]} \frac{\text{up} \quad \text{dn} \quad \text{up}}{-\frac{3}{2} \quad -\frac{5}{4} \quad -1 \quad -\frac{1}{2} \quad \underbrace{0}_{\frac{1}{2}} \quad 1 \quad \frac{5}{4} \quad \frac{3}{2}}$

IV f) The coordinates are listed in IV b.

IV g)

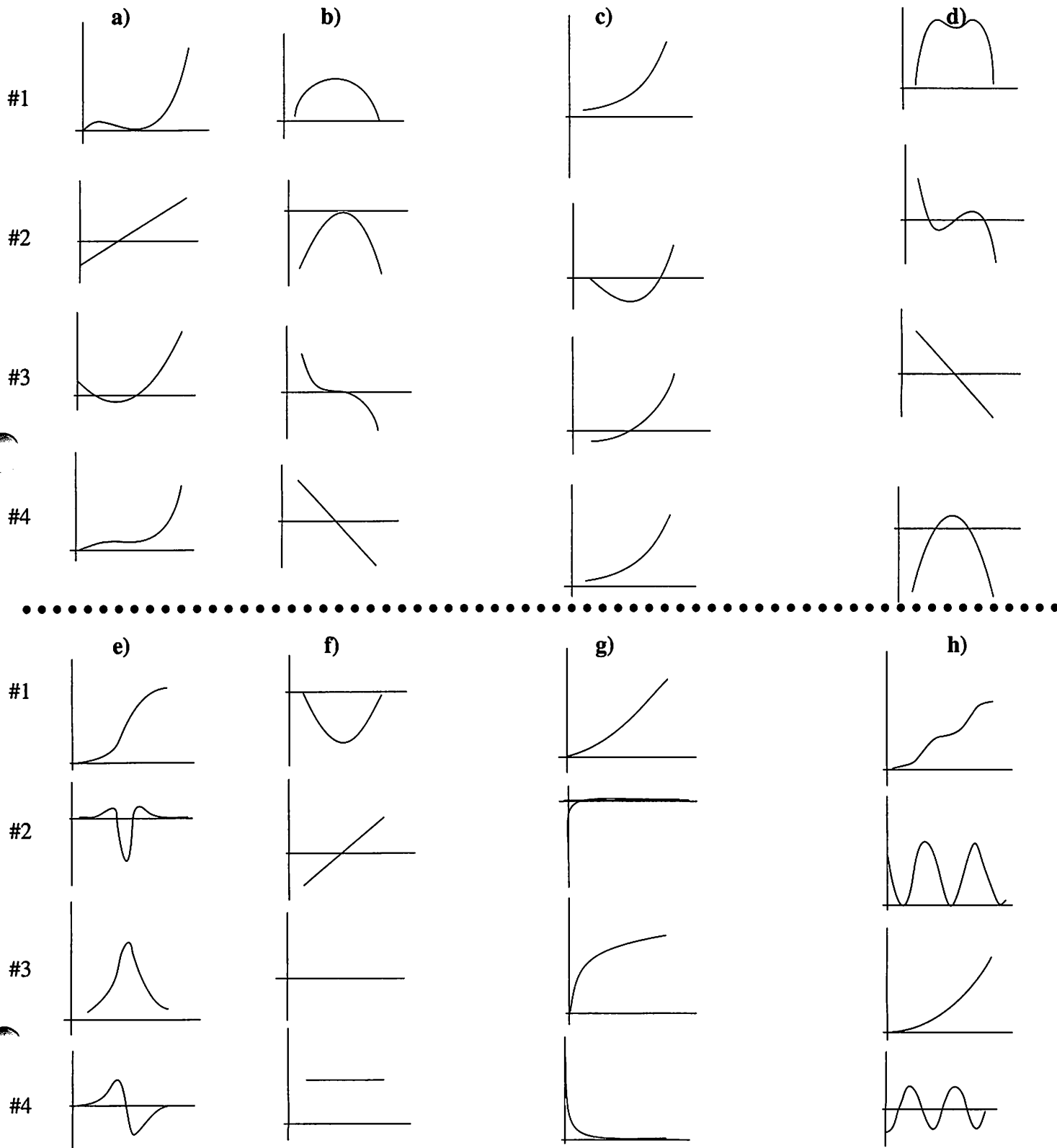


Calculus

$F f' f''$ (un)ctions

J Mumaugh

Each of the following columns contains four related functions in random order, the graphs of $f(x)$, $f'(x)$, $f''(x)$, and the function $F(x)$ (whose derivative is $f(x)$ ie. $F'(x) = f(x)$). Label each graph with the correct function.



Summations & Notation

7.3
p. 262-267

Define: Summation Notation.

Let $a_1, a_2, a_3, \dots, a_n$ be numbers then

$$a_1 + a_2 + a_3 + \dots + a_n = \sum_{i=1}^n a_i$$

Annotations:
 - n : Ending index
 - a_i : Term generator
 - $i=1$: Beginning index
 - i : The index

eg $1^2 + 2^2 + 3^2 + 4^2 = \sum_{i=1}^4 i^2$

Annotations:
 - i : The index or index letter
 - 4 : END - Begin + 1 = # of terms

So $\sum_{i=1}^4 i^2 = 1 + 4 + 9 + 16 = 30$

eg $\sum_{i=1}^3 2^i = 2^1 + 2^2 + 2^3 = 2 + 4 + 8 = 14$

eg $\sum_{j=2}^4 \frac{1}{j} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{6}{12} + \frac{4}{12} + \frac{3}{12} = \frac{13}{12}$

Rule #1 $\sum_{i=1}^N c a_i = c a_1 + c a_2 + \dots + c a_N =$

$$c(a_1 + a_2 + \dots + a_N) = c \sum_{i=1}^N a_i$$

id

a constant factor moves across the $\sum$

Rule #2 $\sum_{i=1}^N (a_i + b_i) = a_1 + b_1 + a_2 + b_2 + \dots + a_N + b_N =$

$$a_1 + a_2 + \dots + a_N + b_1 + b_2 + \dots + b_N = \sum_{i=1}^N a_i + \sum_{i=1}^N b_i$$

INSTRUCT students to read the section on their own after lecture. Let them figure out actual continues.

eg $\sum_{i=1}^5 3 + 0x_i = \sum_{i=1}^5 3 = 3 + 3 + 3 + 3 + 3 = 15$

figure out

Rule 3 $\sum_{i=1}^N C = NC.$

"Telescoping" sums, then we discuss them briefly. They are a necessary part of the definite S proof. --

omit
eg $\sum_{i=2}^6 i = 5 \cdot 2 + 5 \cdot 3 + 5 \cdot 4 + 5 \cdot 5 + 5 \cdot 6 = 10 + 15 + 20 + 25 + 30 = 100$

Rule 4

$\sum_{i=1}^m a_i + \sum_{i=m+1}^N a_i = \sum_{i=1}^N a_i$

m is a "dummy variable"

Not Proved
eg. $\sum_{i=1}^N \frac{1}{2} (x_i - x_{i-1})^2 = \frac{1}{2} (x_1 - x_0)^2 + \frac{1}{2} (x_2 - x_1)^2 + \dots + \frac{1}{2} (x_N - x_{N-1})^2$

eg, write in summation notation:

$(\frac{1}{4})^2 (\frac{1}{2}) + (\frac{3}{4})^2 \frac{1}{2} + (\frac{5}{4})^2 \frac{1}{2} + (\frac{7}{4})^2 \frac{1}{2} + (\frac{9}{4})^2 \frac{1}{2} + (\frac{11}{4})^2 \frac{1}{2}$

1. There are 6 summands $\therefore \sum_{i=1}^6$

2. $\frac{1}{2}$ is constant $\therefore \frac{1}{2} \sum_{i=1}^6$

3. The 4 in the denom is constant $\therefore \frac{1}{2} \sum_{N=1}^6 (\frac{\quad}{4})^2$
and each term is squared

4. Numerator changes as odd # $\therefore 2N-1 \Rightarrow \frac{1}{2} \sum_{N=1}^6 (\frac{2N-1}{4})^2$

5. Pulling the constant out $\frac{1}{2} \cdot \frac{1}{16} \sum_{N=1}^6 (2N-1)^2 =$

$\frac{1}{32} \sum_{N=1}^6 (2N-1)^2$

$$\textcircled{1} \text{ a) } \sum_{i=1}^3 i = 1+2+3=6$$

$$\text{b) } \sum_{i=1}^4 2i = 2+4+6+8=20$$

$$\text{c) } \sum_{d=1}^3 d^2 = 1+4+9=14$$

$$\textcircled{2} \text{ a) } \sum_{i=2}^4 i^2 = 4+9+16=29$$

$$\text{b) } \sum_{j=2}^4 j^2 = 29 \quad 4+9+16$$

$$\text{c) } \sum_{i=1}^3 i^2 + i = 1+1+4+2+9+3=20$$

$$\textcircled{3} \text{ a) } \sum_{i=1}^4 1^i = 1+1+1+1=4$$

$$\text{b) } \sum_{k=2}^6 (-1)^k = 1-1+1-1+1=1$$

$$\text{c) } \sum_{j=1}^{150} 3 = 3 \cdot 150 = 450$$

$$4. \text{ a) } 1+2+\dots+(n-1)+n = \sum_{i=1}^n i$$

$$\text{b) } 1^2+2^2+3^2+\dots+10^2 = \sum_{i=1}^{10} i^2$$

$$\text{c) } 1+3+5+7+9+11+13 = \sum_{i=1}^7 (2i-1)$$

$$5a) 1 + 2 + 2^2 + 2^3 + \dots + 2^{100} = \sum_{i=0}^{100} 2^i$$

$$b) x^3 + x^4 + x^5 + x^6 + x^7 = \sum_{i=3}^7 x^i$$

$$c) \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{102} = \sum_{i=3}^{102} \frac{1}{i}$$

$$d) \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{100} = \sum_{i=2}^{100} \frac{1}{i}$$

$$e) \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} = \sum_{i=1}^5 \frac{1}{2n+1}$$

$$f) \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots + \frac{1}{(101)^2} = \sum_{i=1}^{51} \frac{1}{(2n-1)^2}$$

$$\begin{aligned} 2n-1 &= 101 \\ 2n &\geq 102 \\ n &= 51 \end{aligned}$$

$$6a) \sum_{i=1}^8 x_i^2 (x_i - x_{i-1})$$

$$b) \sum_{i=1}^N x_i^5 (x_i - x_{i-1})$$

$$7. \sum_{i=1}^3 a_i - b_i = \sum_{i=1}^3 a_i - \sum_{i=1}^3 b_i$$

$$a) a_1 - b_1 + a_2 - b_2 + a_3 - b_3 = a_1 + a_2 + a_3 - (b_1 + b_2 + b_3) = \sum_{i=1}^3 a_i - \sum_{i=1}^3 b_i$$

$$b) \sum_{i=1}^2 a_i b_i \neq \sum_{i=1}^2 a_i \sum_{i=1}^2 b_i$$

$$a_1 b_1 + a_2 b_2 \neq (a_1 + a_2)(b_1 + b_2)$$

$$a_1 b_1 + a_2 b_2 \neq a_1 b_1 + a_2 b_2 + \underbrace{a_2 b_1 + a_1 b_2}_{\text{Extra}}$$

$$\therefore \sum_{i=1}^3 \left(\sum_{j=1}^3 b_j \right) a_i = \sum_{j=1}^3 \left(\sum_{i=1}^3 a_i \right) b_j$$

$$\sum_{i=1}^3 (b_1 a_i + b_2 a_i + b_3 a_i) =$$

$$b_1 a_1 + b_1 a_2 + b_1 a_3 + b_2 a_1 + b_2 a_2 + b_3 a_2 + b_3 a_1 + b_3 a_2 + b_3 a_3 =$$

$$b_1 a_1 + b_2 a_1 + b_3 a_1 + b_2 a_2 + b_1 a_2 + b_3 a_2 + a_1 (b_1 + b_2 + b_3) + \text{etc.}$$

$$a_1 \sum_{j=1}^3 b_j + a_2 \sum_{j=1}^3 b_j + a_3 \sum_{j=1}^3 b_j =$$

$$\sum_{j=1}^3 \left(\sum_{i=1}^3 a_i \right) b_j$$

$b_n - b_0$

Telescoping sum

8. $\sum_{i=1}^{100} (2^i - 2^{i-1})$

$$2^{100} - 2^0 = 2^{100} - 1$$

b) $\sum_{i=2}^{100} \left(\frac{1}{i} - \frac{1}{i-1} \right) = \sum_{i=2}^{100} \frac{i-1-i}{i(i-1)} = \sum_{i=2}^{100} \frac{-1}{i(i-1)}$

$$b_1 = 2, b_0 = 1, b_N = \frac{1}{100}$$

$$b_n - b_0 = \frac{1}{100} - 1 = -\frac{99}{100}$$

$$8c) \sum_{i=1}^{50} \left(\frac{1}{2i+1} - \frac{1}{2(i-1)+1} \right)$$

$$b_1 = \frac{1}{3} \quad b_0 = \frac{1}{1} \quad b_n = \frac{1}{101}$$

$$\frac{1}{101} - 1 = -\frac{100}{101}$$

$$[0, 4]$$

$$a) \sum_{i=1}^N (x_i^3 - x_{i-1}^3) =$$

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 \\ | & | & | & | & | \\ x_0 & x_1 & x_2 & x_3 & x_4 \end{array}$$

$$x_N^3 - x_0^3 = 64 - 0 = 64$$

$$b) \sum_{i=1}^N x_i^2 - x_{i-1}^2 \Rightarrow x_N^2 - x_0^2 = 16 - 0 = 16$$

$$c) \sum_{i=1}^N 5x_i - 5x_{i-1} \Rightarrow 5x_N - 5x_0 = 20$$

(10) skip

(11) follow directions very easy. 11c 5050.

$$(12) \sum_{i=1}^N c^{N-i} i^{-1}$$

$$(13) d \sum_{i=1}^N (2i-1) = n^2$$

$$2i+1 = (i+1)^2 - i^2$$

skip a)

$$i^2 + 2i + 1 - i^2 = 2i + 1$$

Wait
Quiz.

$$b) b_N - b_0 \Rightarrow (N+1)^2 -$$

2nd DAY
of
class

$$\textcircled{a} \sum_{i=1}^{100} 2^i - 2^{i-1} =$$

$$\sum_{i=1}^{100} 2^i - 2^{i-1}$$

4)

$$2^1 - 2^0$$

$$2^2 - 2^1$$

$$2^3 - 2^2$$

$$2^{100} - 2^0 = 2^{100} - 1$$

$$b_n - b_0 =$$

b_0 is Left Edge
of 1st SECTION

5)

$$\sum_{i=2}^{100} \frac{1}{i} - \frac{1}{i-1}$$

$$\frac{1}{2^0} - \frac{1}{2^1}$$

$$\frac{1}{2^1} - \frac{1}{2^2}$$

b_n is Right Edge of
LAST SECTION

$$\frac{1}{2} - \frac{1}{1}$$

$$\frac{1}{3} - \frac{1}{2}$$

same pattern

$$\frac{1}{100} - 1 = -\frac{99}{100}$$

$$\textcircled{c} \sum_{i=1}^{50} \left(\frac{1}{2i+1} - \frac{1}{2(i-1)+1} \right)$$

The interval is

$$2^{100} - 1 \text{ wide}$$

thus $b_n - b_0 =$ Length of

$$\frac{1}{101} - 1 = -\frac{100}{101}$$

The interval

each a_i is each term
is a section is $2^i - 2^{i-1}$
is a particular section.

(13) The Square is broken by the Solid lines.
In the smallest Square we have 1 square.
In the next we add 3, the next adds 5 and
the last adds 7 squares.

we have a 4×4 Square which has 16
Squares. So we conclude $1 + 3 + 5 + 7 = 16$.

b) It appears that the sum of the 1st N
positive odd integers is N^2 .

c) Since $2i+1$ is any positive integer ($N \geq 0$)
so i^{th} positive integer would be $(2i+1)$.

$$d) \sum_{i=0}^{N-1} 2i+1$$

(14) $2i+1 = (i+1)^2 - i^2$
a) $= i^2 + 2i + 1 - i^2 = 2i + 1$

b) $2i-1 = i^2 - (i-1)^2$
 $i^2 - (i^2 - 2i + 1)$
 $i^2 - i^2 + 2i - 1$
 $= 2i - 1$

$$\sum_{i=0}^N i^2 - (i-1)^2 = N^2 - 0^2 = N^2$$

$$\sum_{i=0}^{N-1} (i+1)^2 - i^2 = N^2$$

compare answer
in 13 B

$$(15) \quad a) \quad i = \frac{[(i+1)^2 - i^2 - 1]}{2}$$

$$\frac{i^2 + 2i + 1 - i^2 - 1}{2} = i$$

$$b) \quad \sum_{i=1}^N i = \frac{(N+1)^2}{2} - \frac{1}{2} - \frac{N}{2}$$

$$\frac{1}{2} \sum_{i=1}^N [(i+1)^2 - i^2 - 1] = \sum_{i=1}^N \frac{(i+1)^2 - i^2}{2} - \sum_{i=1}^N \frac{1}{2}$$

$$= \sum_{i=1}^N \frac{(i+1)^2}{2} - \frac{i^2}{2} - \sum_{i=1}^N \frac{1}{2}$$

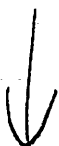
Telescoping Sum. using example 5

$$= \frac{(N+1)^2}{2} - \frac{1}{2} - \frac{N}{2}$$

$$c) \quad \sum_{i=1}^{100} i = \frac{(100+1)^2}{2} - \frac{1}{2} - \frac{100}{2}$$

$$\frac{10201}{2} - \frac{1}{2} - \frac{100}{2} = \frac{10100}{2} = 5050.$$

(16)



$$16 \quad (i+1)^3 - i^3 = 3i^2 + 3i + 1$$

$$a) \quad \cancel{i^3} + 3i^2 + 3i + 1 - \cancel{i^3} = 3i^2 + 3i + 1 = 3i^2 + 3i + 1$$

$$\begin{array}{c} 1 \\ 1 \ 2 \ 1 \\ 1 \ 3 \ 3 \ 1 \end{array}$$

$$\sum_{i=1}^N (i+1)^3 - i^3 = (N+1)^3 - 1$$

$$b) \quad 3 \sum_{i=1}^N i^2 + 3 \sum_{i=1}^N i + \sum_{i=1}^N 1 = (N+1)^3 - 1$$

$$\sum_{i=1}^N (3i^2 + 3i + 1) = \sum_{i=1}^N ((i+1)^3 - i^3) = (N+1)^3 - 1$$

$$c) \quad \sum_{i=1}^N 1 = N$$

$$d) \quad \sum_{i=1}^N i = \frac{(N+1)^2}{2} - \frac{1}{2} - \frac{N}{2} \quad (\text{from ex 15b})$$

$$E) \quad 3 \sum_{i=1}^N i^2 + 3 \sum_{i=1}^N i + \sum_{i=1}^N 1 = (N+1)^3 - 1$$

$$3 \sum_{i=1}^N i^2 = (N+1)^3 - 1 - 3 \sum_{i=1}^N i - \sum_{i=1}^N 1$$

$$\sum_{i=1}^N i^2 = \frac{(N+1)^3}{3} - \frac{1}{3} - \left[\frac{(N+1)^2}{2} - \frac{1}{2} - \frac{N}{2} \right] - \frac{N}{3}$$

$$= \frac{2(N+1)^3 - 2 - 3(N+1)^2 + 3 + 3N - 2N}{6}$$

$$= \frac{2(N+1)^3 - 3(N+1)^2 + N + 1}{6}$$

$$= \frac{(N+1)(2(N+1)^2 - 3(N+1) + 1)}{6}$$

$$= \frac{(N+1)(2N^2 + 4N + 2 - 3N - 3 + 1)}{6}$$

$$\downarrow$$

$$E) = \frac{(N+1)(2N^2+N)}{6}$$

$$= \frac{N(N+1)(2N+1)}{6}$$

$$F) \quad N=5 \quad \sum_{i=1}^5 i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$$

$$\frac{5(5+1)(5 \cdot 2 + 1)}{6}$$

$$\frac{5(6)(11)}{6} = 55 \quad \text{it checks,}$$

Summary of Pblms.

Converting to telescoping sums is difficult but can be done. I still am not satisfied with my understanding of these methods, But that was then, I've got it now -

Explained in lecture on pages 255-261

254 #6

$$d) y = \frac{1}{3(2x+3)} - \frac{1}{9} \ln \left(\frac{2x+3}{x} \right)$$

$$y = \frac{1}{3} (2x+3)^{-1} - \frac{1}{9} \ln(2x+3) + \frac{1}{9} \ln x$$

$$y' = -\frac{2}{3} (2x+3)^{-2} - \frac{2}{9(2x+3)} + \frac{1}{9x}$$

$$e) y = \frac{1}{5} \ln(\csc 5x - \cot 5x)$$

$$y = \frac{1}{5} \left[\ln \left(\frac{1 - \cos 5x}{\sin 5x} \right) \right] = \frac{1}{5} \left[\ln(1 - \cos 5x) - \ln \sin 5x \right]$$

$$y' = \frac{1}{5} \left[\frac{5 \sin 5x}{(1 - \cos 5x)} - \frac{5 \cos 5x}{\sin 5x} \right] =$$

$$\frac{\sin^2 5x - \cos^2 5x + \cos^2 5x}{(1 - \cos 5x) \sin 5x} = \frac{1 - \cos 5x}{(1 - \cos 5x) \sin 5x} =$$

$$\csc 5x$$

$$f) y = x \tan^{-1}(2x) - \frac{1}{4} \ln(1 + 4x^2)$$

$$y' = \frac{2x}{1 + 4x^2} + \tan^{-1}(2x) - \frac{2x}{4(1 + 4x^2)} = \tan^{-1} 2x$$

$$\begin{aligned} \text{6. a)} \quad y &= \sqrt{1 + \cot^2 3x} \\ y' &= \frac{1 \cdot 3 \cot^2 3x}{2 \sqrt{1 + \cot^2 3x}} \\ y &= \frac{-3 \cot^2 3x}{2 \sqrt{1 + \cot^2 3x}} \end{aligned}$$

$$\begin{aligned} \frac{\cos}{\sin} \\ \frac{-\sin^2 x + 3 \cos^2 x}{\sin^2 x} \\ = \frac{1}{\sin^2 x} = \csc^2 x \end{aligned}$$

$$\text{b)} \quad y = \frac{3x}{8} + \frac{3x \sin 4x}{32} + \frac{\cos^3 2x \sin 2x}{8}$$

$$y' = \frac{3}{8} + \frac{3x}{32} \cdot \sin 4x + \frac{1}{8} [\cos^3 2x \cdot \sin 2x]$$

$$y' = \frac{3}{8} + \frac{3 \sin 4x}{32} + \frac{4 \cdot 3x \cos 4x}{3 \cdot 8}$$

$$\frac{1}{8} \left[2(-3) \sin 2x \cos^2 2x \sin 2x + 2 \cos^3 2x \cos 2x \right]$$

$$\begin{aligned} y' &= \frac{3}{8} + \frac{3 \sin 4x}{32} + \frac{3x}{8} \cos 4x + \frac{1}{8} \left[-6 \sin^2 2x \cos^2 2x + 2 \cos^4 2x \right] \\ &= \frac{3}{8} + \frac{3}{32} \sin 4x + \frac{3x}{8} \cos 4x - \frac{3}{4} \sin^2 2x \cos^2 2x + \frac{1}{4} \cos^4 2x \end{aligned}$$

$$\text{c)} \quad y = \frac{3}{8(2x+3)^2} - \frac{1}{4(2x-3)} = \frac{3}{8} (2x+3)^{-2} - \frac{1}{4} (2x-3)^{-1}$$

$$\begin{aligned} u &= 2x+3 \\ du &= 2 \end{aligned}$$

$$z = \frac{1}{u^2} = u^{-2}$$

$$\frac{dz}{du} = -\frac{2}{u^3}$$

$$y' = \frac{3}{8} \left[(-2)(2x+3)^{-3} (2) \right] +$$

$$y' = \frac{3}{8} \left[-4(2x+3)^{-3} \right]$$

$$y' = -\frac{3}{2} (2x+3)^{-3}$$

254 #6

$$d) y = \frac{1}{3(2x+3)} - \frac{1}{9} \ln \left(\frac{2x+3}{x} \right)$$

$$y = \frac{1}{3} (2x+3)^{-1} - \frac{1}{9} \ln(2x+3) + \frac{1}{9} \ln x$$

$$y' = -\frac{2}{3} (2x+3)^{-2} - \frac{2}{9(2x+3)} + \frac{1}{9x}$$

$$e) y = \frac{1}{5} \ln(\csc 5x - \cot 5x)$$

$$y = \frac{1}{5} \left[\ln \left(\frac{1 - \cos 5x}{\sin 5x} \right) \right] = \frac{1}{5} \left[\ln(1 - \cos 5x) - \ln \sin 5x \right]$$

$$y' = \frac{1}{5} \left[\frac{5 \sin 5x}{(1 - \cos 5x)} - \frac{5 \cos 5x}{\sin 5x} \right] =$$

$$\frac{\sin^2 5x - \cos^2 5x + \cos^2 5x}{(1 - \cos 5x) \sin 5x} = \frac{1 - \cos 5x}{(1 - \cos 5x) \sin 5x} =$$

$$\csc 5x$$

$$f) y = x \tan^{-1}(2x) - \frac{1}{4} \ln(1 + 4x^2)$$

$$y' = \frac{2x}{1 + 4x^2} + \tan^{-1}(2x) - \frac{2x}{4(1 + 4x^2)} = \tan^{-1} 2x$$

#7 c) $3x^2 + 2x - 1 + \frac{5}{x} + \frac{3}{x^2} \Rightarrow x^3 + x^2 - x + 5\ln x - \frac{3}{x}$

b) $\frac{x^2}{x^3+1} \Rightarrow \frac{1}{3} \ln(x^3+1)$

c) $\frac{x^3+1}{x^2} = x + \frac{1}{x^2} = x + x^{-2} \Rightarrow \frac{1}{2}x^2 - \frac{1}{x}$

d) $e^{3x} \Rightarrow \frac{1}{3} e^{3x}$

e) $\sin 2x \Rightarrow -\frac{1}{2} \cos 2x$

f) $x e^x \Rightarrow (x-1)e^x$

⑧ $y' = t^2$

⑨

≤ 8 stars

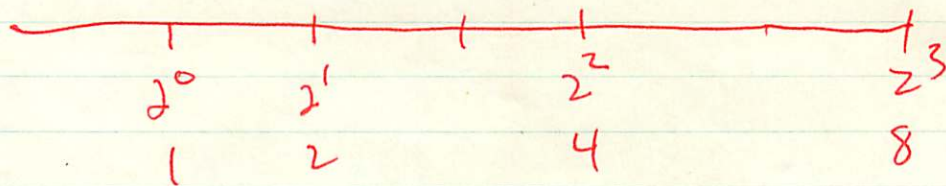
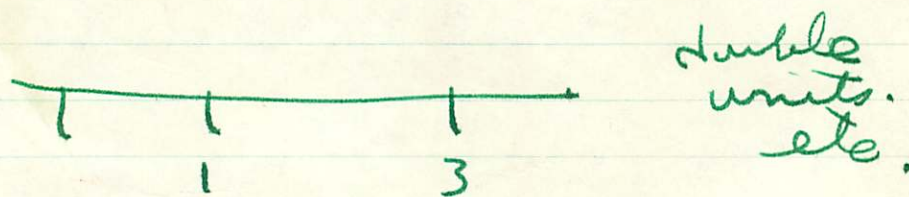
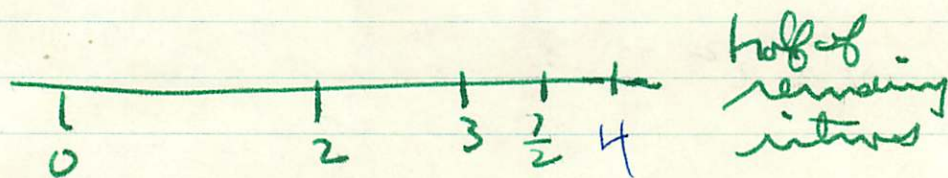
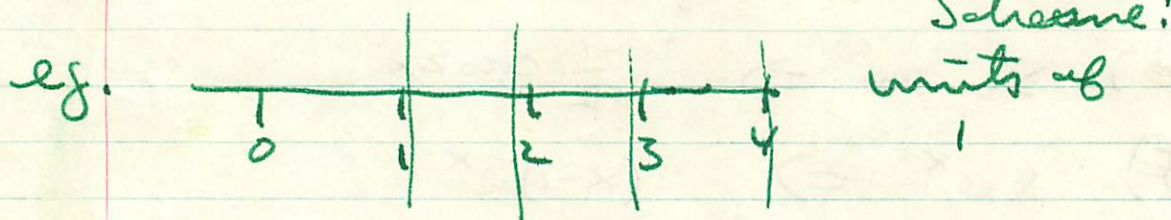
above stars etc.

1 5 5 1 0 6 5 1

* think of "partition"

as the "Scheme used to break up
an interval".

Scheme!



Students don't need to copy notes from the boards until the definition on 3rd link page.

DAY 3

7.2
18255-261
lecture.

General Solutions to the 4 problems.

1. Notice in each the interval $[0, 3]$ was cut into N smaller parts by $n+1$ numbers i.e. $x_0, x_1, \dots, x_N$. Each $x_i - x_{i-1}$ is called a **SECTION** of the interval.



we call $i = 1, 2, \dots, N$ the **PARTITION** of $[a, b]$.

2. END pts of a **SECTION** are x_i on the right and x_{i-1} on the left.

3. Let $\bar{x}_i$ be the sampling pt. in problem 1A on HW it was mid pt of section, 1b it was x_i and in 1d it was x_{i-1} . Regardless this is true $x_{i-1} \leq \bar{x}_i \leq x_i$ **ie sample pt is contained in section**.
4. The estimates made end up with:

height width

$$A \left(\bar{x}_1^2 (x_1 - x_0) + \bar{x}_2^2 (x_2 - x_1) + \dots + \bar{x}_N^2 (x_N - x_{N-1}) \right)$$

ie Sample pt multiplied times length of the section

where A is the constant built into the original problem. ie problem 1 $A=1$, problem 2 $A=5$, problem 3 $A=8$ problem 4 $A=1$.

5. Our under and over estimating sums are then:

under: $x_0^2 (x_1 - x_0) + x_1^2 (x_2 - x_1) + \dots + x_{N-1}^2 (x_N - x_{N-1})$

over: $x_1^2 (x_1 - x_0) + x_2^2 (x_2 - x_1) + \dots + x_N^2 (x_N - x_{N-1})$

Next turn to pg 62. the lemma.

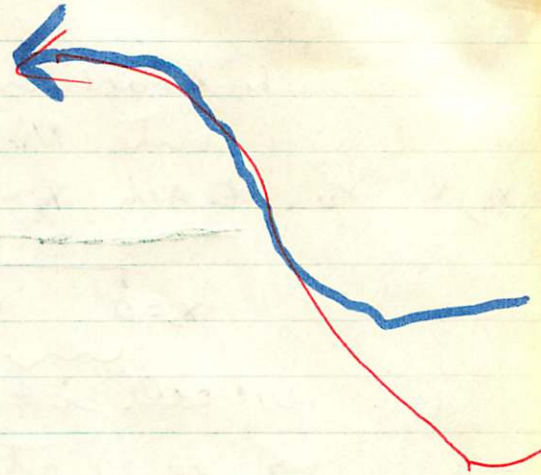
height is $f(\bar{x}_i)$

using left end as sample

using right end as sample

ΣTAT

we choose $n=2$ because
our function is $y=x^2$



*

no/TS22

no/TS22

→

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$$(N+1)d^N(c-d) < c^{N+1} - d^{N+1} < (N+1)c^N(c-d) \quad \text{from Pg 62}$$

$c > d$

$$\div (N+1)$$

$$d^N(c-d) < \frac{c^{N+1} - d^{N+1}}{N+1} < c^N(c-d)$$

Since our function is x^2 ; we let $N=2$.

$$d^2(c-d) < \frac{c^3}{3} - \frac{d^3}{3} < c^2(c-d)$$

$$c = x_1, d = x_0$$

$$x_0^2(x_1 - x_0) < \frac{x_1^3}{3} - \frac{x_0^3}{3} < x_1^2(x_1 - x_0)$$

$$c = x_2, d = x_1$$

$$x_1^2(x_2 - x_1) < \frac{x_2^3}{3} - \frac{x_1^3}{3} < x_2^2(x_2 - x_1)$$

$$c = x_3, d = x_2$$

$$x_2^2(x_3 - x_2) < \frac{x_3^3}{3} - \frac{x_2^3}{3} < x_3^2(x_3 - x_2)$$

add
these
down
to get

$$c = x_N, d = x_{N-1}$$

$$x_{N-1}^2(x_N - x_{N-1}) < \frac{x_N^3}{3} - \frac{x_{N-1}^3}{3} < x_N^2(x_N - x_{N-1})$$

ADD the
3 columns

$$\sum_{i=1}^N x_{i-1}^2(x_i - x_{i-1}) < \sum_{i=1}^N \left(\frac{x_i^3}{3} - \frac{x_{i-1}^3}{3} \right) < \sum_{i=1}^N x_i^2(x_i - x_{i-1})$$

Notice the middle column "telescopes" to $\frac{x_N^3}{3} - \frac{x_0^3}{3}$

in our example problems $x_N = 3$ and $x_0 = 0$

so this middle becomes $\frac{27}{3} - 0 = 9$. The middle is fixed at nine. The left and right columns will each approach that fixed value of 9.

Generally

Let S be some region in the plane whose area is to be found. Let L be a line in the plane which we call the x -axis. Each line in the plane $\perp$ to L meets S is what is called a CROSS SECTION. The length of the cross section is $C(x)$.

The interval is defined to have left end point $x=a$ and right end point $x=b$.

This interval is divided into sections by some means ^(or rule) called the partition. The largest (longest) of the sections is called the mesh. As the number of sections increase the mesh is forced to become smaller.

In the string problem, we think of density of each section to be almost constant so we have the mass of that section. we continue the process, adding the mass of each section to get the mass entire interval.

In the Speed problem, when the mesh is small (if the number of partitions is large) the speed is almost constant so the product of the almost constant speed times the time gives the distance over that ^{very short} period of time. we total all such distances to get the total distance.

In the Tent problem we "add up" a bunch of areas to give us volume.

Students to start with notes.

Definition

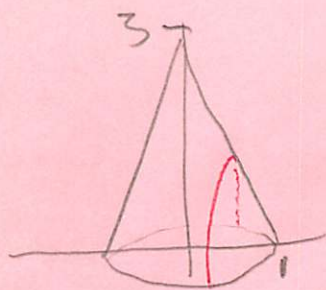
this phrase translates into ~~on~~ mesh $\rightarrow 0$ in limit.

If f is a fcn defined over $[a, b]$ and the sums $\sum_{i=1}^N f(x_i)(x_i - x_{i-1})$ approach a number as the mesh shrinks toward zero that number is called the definite integral of f over $[a, b]$

The definite integral of f from a to b is written: $\int_a^b f(x) dx$

where $\int \Rightarrow$ sum dx refers to the length of the mesh and $f(x)$ refers to cross sections.

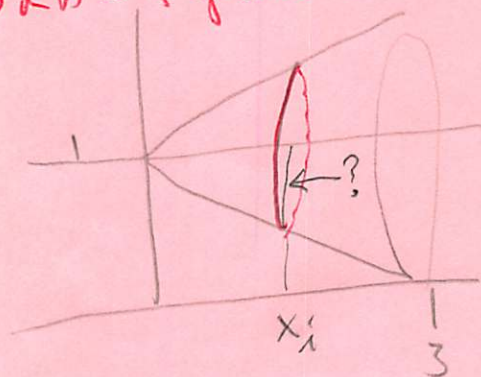
Remember: The definite integral is a Number



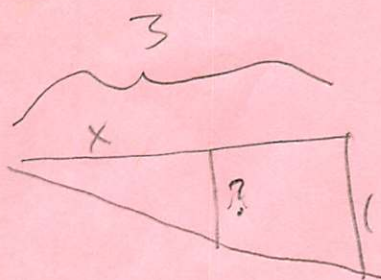
Find The Volume of a right circular cone of height 3 and radius 1.

This version yields a ~~portion of~~ hyperbola. Its area is not easily identified

DRAW Figure as below so integral will be simplest



This version yields a circle. we know the area is πR^2



Area of typical cross section is πr^2

Similar Δ s

$$\frac{r}{x} = \frac{1}{3}$$

$$r = \frac{x}{3}$$

$$\int_0^3 \pi \left(\frac{x}{3}\right)^2 dx =$$

$$\int_0^3 \frac{\pi}{9} x^2 dx =$$

$$\frac{\pi}{9} \int_0^3 x^2 dx =$$

$$\frac{\pi}{9} \cdot 9 =$$

$$\pi$$

Now Hand out Reminders

$c > d \geq 0$

$0 \leq d < c$ 256-276 Lecture

~~Lemma $c^{N+1} - d^{N+1} < (N+1)c^N(c-d)$ and~~

~~$c^{N+1} - d^{N+1} > (N+1)d^N(c-d)$ which~~

~~becomes:~~

Begin Here $\Rightarrow (N+1)d^N(c-d) < c^{N+1} - d^{N+1} < (N+1)c^N(c-d) \div (N+1)$

$$d^N(c-d) < \frac{c^{N+1}}{N+1} - \frac{d^{N+1}}{N+1} < c^N(c-d)$$

when $N=2$ this becomes:

$$d^2(c-d) < \frac{c^3}{3} - \frac{d^3}{3} < c^2(c-d) \text{ compare}$$

using #5
on previous
web pageif $c = x_1$ & $d = x_0$ then $c = x_2$ & $d = x_1$, etc.

if each inequality is formed & then we sum each portion of the inequality we get.

(under) $\bar{x}_0^2(x_1 - x_0) + \bar{x}_1^2(x_2 - x_1) + \dots + \bar{x}_{N-1}^2(x_N - x_{N-1}) <$

middle

$$\left(\frac{x_1^3}{3} - \frac{x_0^3}{3} + \frac{x_2^3}{3} - \frac{x_1^3}{3} + \dots + \frac{x_N^3}{3} - \frac{x_{N-1}^3}{3} \right) <$$

Notice $\rightarrow$

$$\bar{x}_1^2(x_1 - x_0) + \bar{x}_2^2(x_2 - x_1) + \dots + \bar{x}_N^2(x_N - x_{N-1})$$

The middle simplifies to $\frac{x_N^3}{3} - \frac{x_0^3}{3}$ because it is a telescoping sum in ourpb/ems $x_0 = 0$ & $x_N = 3$ $\therefore$ the middle becomes 9.

The number of sections gets large

we know that as $N \rightarrow \infty$ our estimates become nearer to the actual quantity. Notice though the middle is fixed at 9. Clearly both the under & over estimates are closing in on 9.

Vocabulary, is getting away from the 4 problem
and talking in general terms. 255-276
lecture

Pf 267 {
Let S be some region in the plane whose area is to be found, let L be a line in the plane which we call x -axis, each line in the plane $\perp$ to L meets S in what is called a cross section. The length of the cross section is $C(x)$, assume that the lines which are $\perp$ to L and meet S intersect L ^{within} an interval. ~~whose~~ ^{the} endpoints ^{of that interval} are $x=a$ and $x=b$.

In Prob 1 L was x axis

$$C(x) = x^2$$

$$a = 0$$

$$b = 3$$

The length of the longest section $x_1 - x_{i-1}$ is called the MESH in the Partition.

Thus in the previous example we let the mesh $\rightarrow 0$, means we increased the number of sections to ∞ .

If we consider the string problem we must think of the mesh being very small then we can say that the density in each section is almost constant. Then all that is required is to sum each of the section's ^{mass} to arrive at the total mass.

In the Speed Problem the same idea occurs, for a short period of time the speed is almost constant so we sum all the short distances to arrive at the total distance driven.

In the tent problem we computed Areas to arrive at Volume.

One important item to be recognized is that if we can describe a small portion, ^{then} we can end up with the total Volume, distance, mass etc.

The tent problem was easy because we could talk about a cross section. The volume of a cylinder would be easy to discuss since its cross section is easy to describe (a circle).

Definition: The Definite integral of a fctn f over an interval $[a, b]$.

If f is a fctn defined over $[a, b]$ and the sums $\sum_{k=1}^n f(x_k)(x_k - x_{k-1})$ approach a number as the mesh shrinks toward zero that number is called the definite integral of f over that $[a, b]$.

The definite integral of f from a to b is simply the integral of f from a to b is written $\int_a^b f(x) dx$.

$\int \Rightarrow$ sum, dx refers to the length of the mesh and $f(x)$ refers to height in cross section.



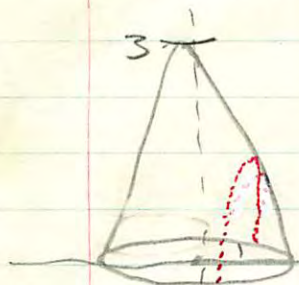
Remember. The definite integral is A Number!!

Students to copy into notes!

In passing, we accept that this definite integral exists and think of it as a number rather than an area exclusively. The reason the distinction is made is because then we can have more than 3 dimensions for a problem. The mathematics is straight forward even if the physical object is not.

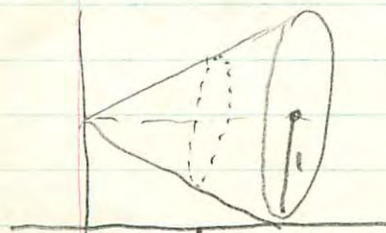
consider: length, height, depth as 3 constraints on an object compared with constraints upon the selling price of some item or service. Clearly there can be more than 3 controllable items. *Cost of Production, sales price, shape of container, color of container, Advertising, the economy*

eg 1 Find the volume of a right circular cone of height 3 feet and radius 1 foot.



this version yields a ^{position of a hyperbola} ~~parabola~~ as a cross section. the area within a ~~hyperbola~~ ^{hyperbola} is not obvious yet.

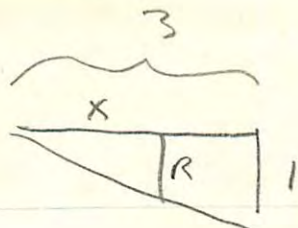
So we draw the figure from a different pt. of view



Now the cross section is obviously a circle & the area is easy to describe.

area of cross section is πR^2
 Notice radius of circle depends on x ,
 we need to re write π in terms of x .





using Similar A.S. 255-276
lecture

$$\frac{x}{3} = \frac{R}{1} \text{ i.e. } R = \frac{x}{3} \text{ so}$$

radius of cross section is $\frac{x}{3}$ so
area of cross section is $\frac{\pi x^2}{9}$

Now we need only "Sum up" all such areas.

constants only can be moved across $\int$ sign.

$$\int_0^3 \frac{\pi x^2}{9} dx = \frac{\pi}{9} \int_0^3 x^2 dx \quad \text{Since } \int \text{ is Just a fancy } \Sigma$$

$$\int_0^3 x^2 dx = 9 \text{ as per our earlier "Sandwiching" problem.}$$

$$\text{thus } \int_0^3 \frac{\pi x^2}{9} dx = \frac{\pi}{9} \cdot 9 = \pi \text{ so}$$

END of 1st day of lecture

the volume of the cone is π cubic feet.

~~fm CRC Book $V = \frac{1}{3} \pi R^2 h$, R = radius of base, h = height. $V = \frac{1}{3} \pi \cdot 1 \cdot 3 = \pi$~~

eg #2 (Pg 272) $0 \leq a \leq b$

General rather than specific is not

compute the definite integral $\int_a^b x^2 dx$ as in our earlier method we partition the interval $[a, b]$ such that

$a_0 = x_0 < x_1 < x_2 \dots < x_n = b$ then we look at the sums $\sum_{i=1}^n X_i^2 (x_i - x_{i-1})$ when the mesh is small

the mesh becomes small as the # of sections (i.e. the value of n) gets large.

as Ireland
Yesterday

255-276

Lecture

the underestimating sum was

$$\sum_{i=1}^N x_{i-1}^2 (x_i - x_{i-1}) \quad \text{Right edge of section}$$

$$\text{Sum was } \sum_{i=1}^N x_i^2 (x_i - x_{i-1}) \quad \text{Left edge of each section}$$

The inequalities we developed in chart 4 are brought in (used when $N=2$)

$$d^3(c-d) \leq \frac{c^3}{3} - \frac{d^3}{3} \leq c^2(c-d)$$

when $x_{i-1} = d$ and $x_i = c$ and we add up all the columns we have

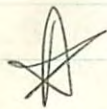
$$\sum_{i=1}^N x_{i-1}^2 (x_i - x_{i-1}) \leq \sum_{i=1}^N \left(\frac{x_i^3}{3} - \frac{x_{i-1}^3}{3} \right) \leq \sum_{i=1}^N x_i^2 (x_i - x_{i-1})$$

The middle sum is a "telescoping sum" it's the "sandwich" the desired sum to

$$\frac{x_N^3}{3} - \frac{x_0^3}{3} \quad \text{Since } x_0 = a, x_N = b$$

we have

$$\frac{b^3}{3} - \frac{a^3}{3}$$



The last chore is to show that ^{both} the two estimating sums indeed approach the same L. The clearest way to show that both approach the same number is to show that their difference approaches zero.

wait as for the μ_j -

wait as for the μ_j -

$$x_{i-1} \leq \bar{x}_i \leq x_i$$

as $x_i - x_{i-1} \rightarrow 0$
 $\bar{x}_i$ measures both x_{i-1} and x_i
at the same time.
and if we want close enough
we let n get even bigger.

$$\sum_{i=1}^N \bar{x}_i^2 (x_i - x_{i-1}) - \sum_{i=1}^N \bar{x}_{i-1}^2 (x_i - x_{i-1}) = ?$$

$$\sum_{i=1}^N (\bar{x}_i^2 - \bar{x}_{i-1}^2) (x_i - x_{i-1}) \leq J \sum_{i=1}^N \bar{x}_i - \bar{x}_{i-1}$$

we know J is the largest section - some fix it at length J

Notice $x_i - x_{i-1}$ is the mesh, call it J for a moment, then (the effort is to remove dependence on i .)

the mesh is so small

that $\bar{x}_i$ is both endpoints at same time.

$$\sum_{i=1}^N (\bar{x}_i^2 - \bar{x}_{i-1}^2) J = J \sum_{i=1}^N \bar{x}_i^2 - \bar{x}_{i-1}^2 =$$

$\lim_{J \rightarrow 0}$

Telescoping
Sums again

$$J(\bar{x}_N^2 - \bar{x}_0^2) =$$

but $\bar{x}_N = b, \bar{x}_0 = a$ so: $J(b^2 - a^2)$

$\lim_{J \rightarrow 0}$

Now as J gets small the product $J(b^2 - a^2)$ also gets very small, indeed, it goes to zero as J goes to zero.

So since the difference of the sums goes to zero, the two sums are both approaching the same sum.

Notice though that this entire proof the mention of area was excluded! as a matter of fact no physical terms were used. They were not necessary since the definite integral is number, it is not restricted to area, mass, distance etc.

Conclusion

for $0 \leq a \leq b$

$$\int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{3}$$

However, since the process is so vivid when area is excluded we use area as our Basis or tool.

eg 3 gives ~~$\int_1^3 f(x) dx$~~ $f(x)$ is
 is a fctn whose value $\forall x \in \mathbb{R}$ is 4, i.e. $f(x) = 4$
 is a constant fctn. Evaluate the given
using the definition of the definite integral
in the interval $[1, 3]$

$x_0 = 1$ $x_n = 3$ The sum $\sum_{i=1}^n f(x_i)(x_i - x_{i-1})$

becomes $\lim_{\text{mesh} \rightarrow 0} \sum_{i=1}^n 4(x_i - x_{i-1})$ because $f(x) = 4 \forall x$

as mesh $\rightarrow 0$ $N \rightarrow \infty$

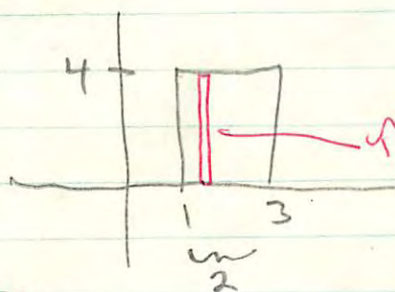
$\lim_{N \rightarrow \infty} 4 \sum_{i=1}^N (x_i - x_{i-1}) = 4(x_N - x_0) =$ using telescoping sum

But the effect of the telescoping sum still remains Thus $4(3 - 1) =$

using our
integral
we get

$\therefore \int_1^3 4 dx = 8$ $4(2) = 8$ is this reasonable?

Same; consider it in terms of area.



The area is obviously 8.

The "typical cross section" is not at either end
 not does it look like it is in the center.

Conclusion

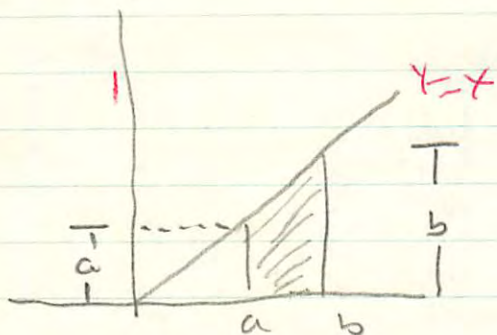
for any constant fctn $\int_a^b c dx = c(b-a)$
 Not worth mentioning.

eg 4 $\downarrow$

eg 4 compute: $\int_a^b f(x) dx$ when $f(x) = x$.

if we consider this as area for convenience we get

without using
with calculator



area of big Δ is $\frac{1}{2} b^2$

area of small Δ is $\frac{1}{2} a^2$.

required area is $\frac{1}{2} b^2 - \frac{1}{2} a^2$.

by using the definitions of the definite integral we also arrive at this number, $\int_a^b x dx$

generally for ANY Positive integer n

$$\int_a^b x^n dx = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$$

putting
not doing
problems

more commonly written

$$\int_a^b x^n dx = \left. \frac{x^{n+1}}{n+1} \right|_a^b = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$$

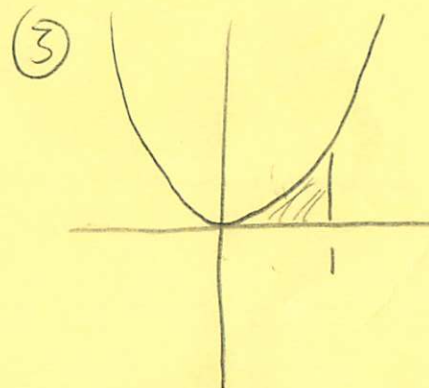
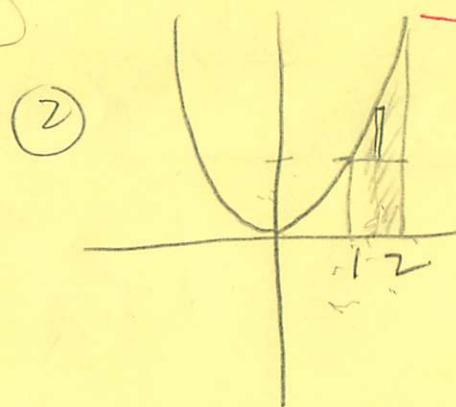
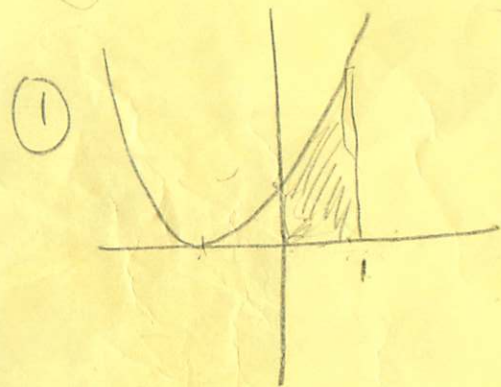
END of 2nd Day of 1st Lecture
(5th day unit)

$$A) \sum_{i=0}^{N-1} \frac{1}{N} \left(\frac{1+i}{N} \right)^2$$

$$B) \sum_{i=1}^N \frac{1}{N} \left(1 + \frac{i}{N} \right)^2$$

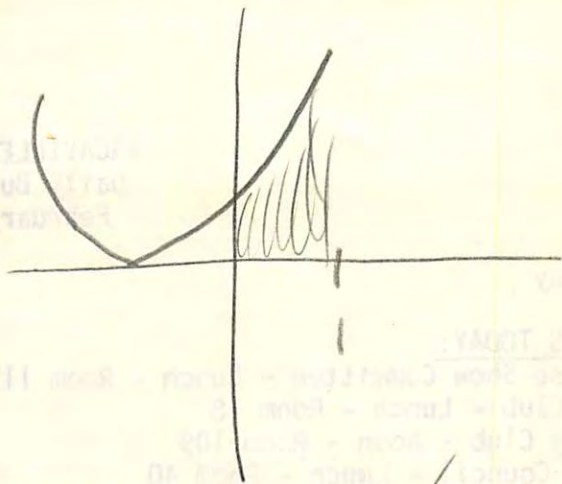
$$C) \sum_{i=1}^N \frac{1}{N} \left(\frac{i}{N} \right)^2$$

$$D) \sum_{i=1}^N \frac{1}{N} \left(1 + \left(\frac{i}{N} \right)^2 \right)$$

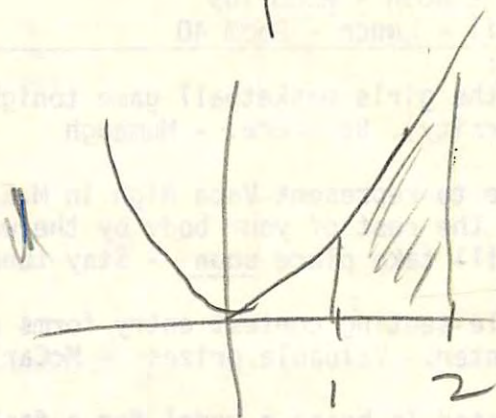


also "suggest" an "E" and a "5" in this "matching."
Don't hint that one maybe matched with more than one other.

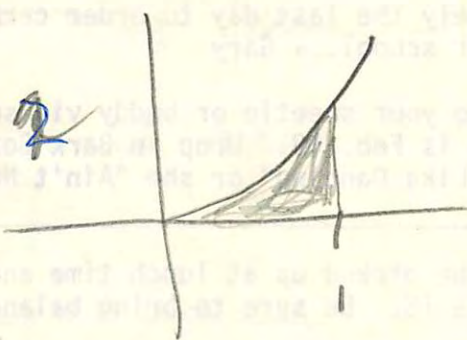
$$a \sum_{i=0}^{n-1} \frac{1}{n} \left(1 + \frac{i}{n}\right)^2$$



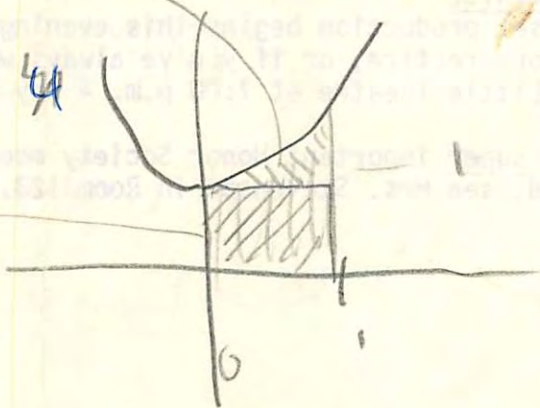
$$b \sum_{i=1}^n \frac{1}{n} \left(1 + \frac{i}{n}\right)^2$$



$$c \sum_{i=1}^n \frac{1}{n} \left(\frac{i}{n}\right)^2$$



$$d \sum_{i=1}^n \frac{1}{n} \left(1 + \left(\frac{i}{n}\right)^2\right)$$



Conclusion
based on plots
11, 12, 13.

- d) $\sum_{i=1}^n f(t_i^*) (t_i - t_{i-1})$ approx gallons during time interval
- e) $\int_a^b f(x) dx$ exact gall entry bble.

276-277

⑤ a) $\int_5^7 3 \, dx = 3(7-5) = 6$

b) $\int_2^8 x \, dx = \frac{8^2}{2} - \frac{2^2}{2} = 32 - 2 = 30$

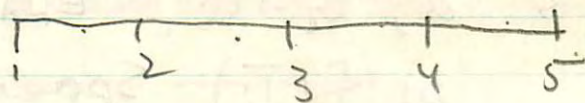
c) $\int_4^6 x^2 \, dx = \frac{6^3}{3} - \frac{4^3}{3} = \frac{216}{3} - \frac{64}{3} = \frac{152}{3}$

⑥ a) $\int_0^1 x^3 \, dx = \frac{1^4}{4} - \frac{0^4}{4} = \frac{1}{4}$

b) $\int_0^1 x^4 \, dx = \frac{1^5}{5} - \frac{0^5}{5} = \frac{1}{5}$

c) $\int_1^2 x^5 \, dx = \frac{2^6}{6} - \frac{1^6}{6} = \frac{64}{6} - \frac{1}{6} = \frac{63}{6} = \frac{21}{2}$

⑦ $\int_1^5 \frac{1}{x} \, dx \quad [1, 5]$



a) left end pts.

$$1(1) + \frac{1}{2}(1) + \frac{1}{3}(1) + \frac{1}{4}(1) = \frac{12 + 6 + 4 + 3}{12} = \frac{25}{12}$$

b) RT end pts

$$\frac{1}{2}(1) + \frac{1}{3}(1) + \frac{1}{4}(1) + \frac{1}{5}(1) = \frac{30 + 20 + 15 + 12}{60} = \frac{77}{60}$$

⑧ $\int_a^b x^2 \, dx = \frac{b^3}{3} - \frac{a^3}{3}$

a) $\int_0^1 x^2 \, dx = \frac{1}{3}$

b) $\int_2^3 x^2 \, dx = \frac{3^3}{3} - \frac{2^3}{3} = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$

c) $\int_1^3 x^2 \, dx = \frac{3^3}{3} - \frac{1^3}{3} = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}$

- ① [1, 6] a) mesh is 1 (length of section)
b) mesh is 2 Longest of all sections
c) mesh is 3

- ② a) $x_i - x_{i-1}$ length of i^{th} section
b) $f(x_i)$ density at sampling pt.
c) $f(x_i)(x_i - x_{i-1})$ approximation of the mass of the i^{th} section
d) $\sum_{i=1}^n f(x_i)(x_i - x_{i-1})$ Approx. mass of entire string.
e) $\int_a^b f(x) dx$ Exact mass of string.

- 3 a) $t_i - t_{i-1}$ Elapsed time of i^{th} time interval
b) $f(t_i)$ Speed at sample pt t_i
c) $f(t_i)(t_i - t_{i-1})$ Approx Dist traveled during i^{th} time interval
d) $\sum_{i=1}^n f(t_i)(t_i - t_{i-1})$ Approx dist traveled
e) $\int_a^b f(t) dt$ Exact dist traveled,

4. a) $t_i - t_{i-1}$ length of time of i^{th} time interval.
b) $f(t_i)$ Rate of flow of water at sample time t_i
c) $f(t_i)(t_i - t_{i-1})$ approx gallons of water which flowed during i^{th} time interval
d) $\sum_{i=1}^n f(t_i)(t_i - t_{i-1})$ approx gallons during time interval
e) $\int_a^b f(t) dt$ exact gall entering lake.

$$⑤ \quad a) \int_5^7 3 \, dx = 3(7-5) = 6$$

$$b) \int_2^8 x \, dx = \frac{8^2}{2} - \frac{2^2}{2} = 32 - 2 = 30$$

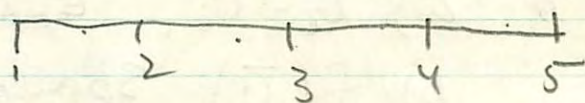
$$c) \int_4^6 x^2 \, dx = \frac{6^3}{3} - \frac{4^3}{3} = \frac{216}{3} - \frac{64}{3} = \frac{152}{3}$$

$$⑥ \quad a) \int_0^1 x^3 \, dx = \frac{1^4}{4} - \frac{0^4}{4} = \frac{1}{4}$$

$$b) \int_0^1 x^4 \, dx = \frac{1^5}{5} - \frac{0^5}{5} = \frac{1}{5}$$

$$c) \int_1^2 x^5 \, dx = \frac{2^6}{6} - \frac{1^6}{6} = \frac{64}{6} - \frac{1}{6} = \frac{63}{6} = \frac{21}{2}$$

$$⑦ \quad \int_1^5 \frac{1}{x} \, dx \quad [1, 5]$$



a) left end pts.

$$1(1) + \frac{1}{2}(1) + \frac{1}{3}(1) + \frac{1}{4}(1) = \frac{12 + 6 + 4 + 3}{12} = \frac{25}{12}$$

b) RT end pts

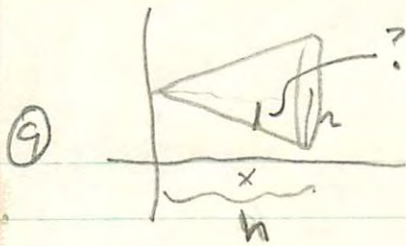
$$\frac{1}{2}(1) + \frac{1}{3}(1) + \frac{1}{4}(1) + \frac{1}{5}(1) = \frac{30 + 20 + 15 + 12}{60} = \frac{77}{60}$$

$$⑧ \quad \int_a^b x^2 \, dx = \frac{b^3}{3} - \frac{a^3}{3}$$

$$a) \int_0^1 x^2 \, dx = \frac{1}{3}$$

$$b) \int_2^3 x^2 \, dx = \frac{3^3}{3} - \frac{2^3}{3} = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$$

$$c) \int_1^3 x^2 \, dx = \frac{3^3}{3} - \frac{1^3}{3} = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}$$



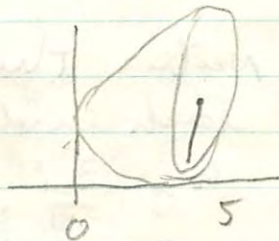
$$\frac{x}{h} = \frac{?}{2} \quad \text{so} \quad ? = \frac{1x}{h}$$

Area of one circle $\Rightarrow \pi \left(\frac{1x}{h}\right)^2$

$$\Sigma \rightarrow \int_0^h \frac{\pi n^2 x^2}{h^2} dx = \frac{\pi n^2}{h^2} \int_0^h x^2 dx$$

$$\frac{\pi n^2}{h^2} \left(\frac{h^3}{3} - \frac{0^3}{3} \right) = \frac{\pi n^2 h}{3}$$

⑩ a)



Typical radius

$$\sqrt{\frac{x}{\pi}}$$

$$\int_0^5 \pi \left(\sqrt{\frac{x}{\pi}} \right)^2 dx = \int_0^5 x dx = \frac{5^2}{2} = \frac{25}{2}$$

10a

10b

⑪

a) $\int_0^2 x^3 dx$

b) $\frac{2^4}{4} - \frac{0^4}{4} = 4$

c) $\int_2^3 x^3 dx$

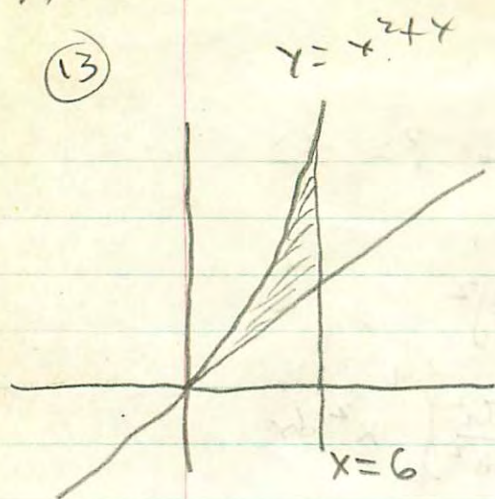
d) $\frac{3^4}{4} - \frac{2^4}{4} = \frac{81}{4} - \frac{16}{4} = \frac{65}{4}$

⑫

Because both are computed in exactly the same manner.

$$L \cdot T \Leftrightarrow L \cdot W$$

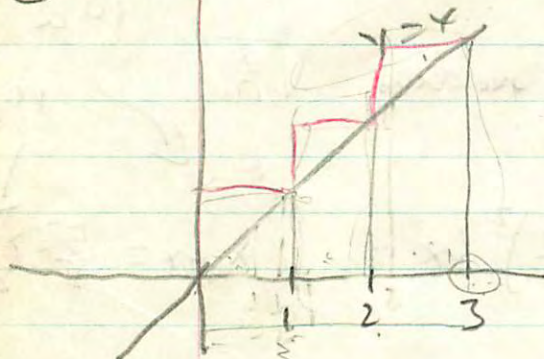
(13)



$$\int_0^6 (x^2 + x) dx - \int_0^6 x dx =$$

$$\int_0^6 x^2 dx = \frac{x^3}{3} \Big|_0^6 = \frac{216}{3} = 72$$

(14)



$$x_0 = 0$$

$$x_1 = \frac{3}{n}$$

$$x_2 = \frac{6}{n}$$

$$x_i = \frac{3i}{n}$$

Nature the location of each endpoint is given by $\frac{3i}{n}$.

14c

$$A_n = \sum_{i=1}^n$$

$$\left(\frac{3i}{n} \right) \left(\frac{3}{n} \right)$$

$$\frac{3}{n} = \frac{\text{length of interval}}{\text{\# of sections}}, \quad \frac{n}{2} (b+1)$$

length of each section
height of each section,

14d

$$A_n = \frac{9}{n^2} \left(\sum_{i=1}^n i \right) = \frac{9}{n^2} \left(\frac{n^2 + n}{2} \right) = \frac{9}{2} \left(1 + \frac{1}{n} \right)$$

14e

$$\lim_{n \rightarrow \infty} \frac{9}{2} \left(1 + \frac{1}{n} \right) = \frac{9}{2}$$

14f

$$\int_0^3 x dx = \frac{x^2}{2} \Big|_0^3 = \frac{9}{2}$$

Since area $\rightarrow \frac{9}{2}$ then the definite integral also goes to $\frac{9}{2}$

$$\textcircled{1} \quad \sum_{i=1}^3 2i^2 = 2 \cdot 1^2 + 2 \cdot 2^2 + 2 \cdot 3^2 = 2 + 8 + 18 = 28$$

$$\text{b) } \sum_{i=1}^{50} [3i^4 - 3(i-1)^4] = \sum_{i=1}^{50} [3i^4 - 3(i^4 - 4i^3 + 6i^2 - 4i + 1)] =$$

$$\sum_{i=1}^{50} [12i^3 - 18i^2 + 12i - 3] \quad \text{telescoping sums.}$$

$$\sum_{i=1}^{50} [3i^4 - 3(i-1)^4] = 3(50)^4 - 3(0)$$

$$3(6,250,000) - 0 =$$

$$18,750,000 - 0 = 18,750,000$$

Notice $b_i = 3i^4$

$$b_{i-1} = 3(i-1)^4$$

$$a_i = b_i - b_{i-1}$$

$$\sum_{i=1}^N a_i = \sum_{i=1}^N b_i - b_{i-1} = b_N - b_0$$

$$\text{c) } \sum_{j=3}^6 5j = 5 \cdot 3 + 5 \cdot 4 + 5 \cdot 5 + 5 \cdot 6$$

$$5(3+4+5+6) = 90$$

$$\textcircled{2} \quad \text{a) } \int_2^5 x^2 dx = \frac{5^3}{3} - \frac{2^3}{3} = \frac{125}{3} - \frac{8}{3} = \frac{117}{3} = 39$$

$$\text{b) } \int_0^2 5 dx = 5(2-0) = 10 \quad \left. \frac{6x^2}{2} \right|_1^3$$

$$\text{c) } \int_1^5 6x dx = 6 \left(\frac{5^2}{2} - \frac{1^2}{2} \right) = 6 \cdot \left(\frac{25}{2} - \frac{1}{2} \right) = 172$$

$$\text{d) } \int_1^2 x^4 dx = \frac{2^5}{5} - \frac{1^5}{5} = \frac{31}{5}$$

3a
$$\sum_{i=1}^3 x_i^3 (x_i - x_{i-1}) =$$

a)
$$x_0^3(x_1 - x_0) + x_1^3(x_2 - x_1) + x_2^3(x_3 - x_2)$$

b) $x_0 = 2 \quad x_1 = 3 \quad x_2 = 3.5 \quad x_3 = 4$

$$8(1) + 27\left(\frac{1}{2}\right) + \frac{49}{4}\left(\frac{1}{2}\right) = \frac{64 + 108 + 49}{8} = \frac{221}{8}$$

~~not required~~

$$\int_2^4 x^3 dx$$

c) Smaller because left end pts were used as sampling pts, and curve is concave up in the interval $x > 0$

Review ex chap 7

Pg 280

$$\textcircled{1} \text{ a) } \int_1^5 x^2 dx = \frac{5^3}{3} - \frac{1^3}{3} = \frac{124}{3}$$

$$\text{b) } \int_2^6 3 dx = 3(6-2) = 12$$

$$\text{c) } \int_1^{10} x dx = \frac{10^2}{2} - \frac{1^2}{2} = \frac{99}{2}$$

$$\text{d) } \int_0^2 x^3 dx = \frac{2^4}{4} = 4$$

$$\textcircled{2} \text{ a) } \int_0^1 x dx \int_0^1 x^2 dx \stackrel{?}{=} \int_0^1 x^3 dx$$

$$\left(\frac{x^2}{2} - 0\right)\left(\frac{x^3}{3} - 0\right) = \frac{x^4}{4} - 0$$

$$\frac{x^5}{6} \neq \frac{x^4}{4}$$

$$\text{b) } \int_0^1 x dx + \int_0^1 x^2 dx = \int_0^1 (x+x^2) dx$$

Yes Pg 263 (of 3)

$$\text{c) } \int_0^1 x^2 dx + \int_1^2 x^2 dx = \int_0^2 x^2 dx$$

Yes, Pg 264

$$\textcircled{3} \int_0^1 \sqrt{1+x^3} dx \quad [0, 1] \quad 50 \text{ Sections.}$$

a) each Section is $\frac{1}{50}$ in length.

$$\text{b) } x_0 = 0 \quad x_1 = \frac{1}{50} \quad x_i = \frac{i}{50}$$

$$\text{c) } X_1 = \frac{1}{100} \quad X_2 = \frac{3}{100} \quad X_i = \frac{2i-1}{100}$$

$$\textcircled{4} \text{ a) } \sum_{j=1}^3 d^j = d + d^2 + d^3 \quad \text{b) } \sum_{k=1}^4 x^k = x + x^2 + x^3 + x^4$$

$$\text{c) } \sum_{i=0}^3 i 2^{-i} = 0 + \frac{1}{2} + \frac{2}{2} + \frac{3}{2} \quad \frac{6}{2} = 3 \quad \text{d) } \sum_{i=2}^5 \frac{i+1}{i} = \frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \frac{6}{5}$$

$$\text{e) } \sum_{i=2}^4 \frac{1}{i} - \frac{1}{i+1} = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$$

$$\text{f) } \sum_{i=1}^4 \sin \frac{i\pi}{4} = \sin \frac{\pi}{4} + \sin \frac{2\pi}{4} + \sin \frac{3\pi}{4} + \sin \pi$$

$$\frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2} + 0 = 1 + \sqrt{2}$$

$$\textcircled{5} \text{ a) } \sum_{i=1}^{100} 2^i - 2^{i-1} \Rightarrow 2^{100} - 2^0 = 2^{100} - 1$$

$$\text{b) } \sum_{i=0}^{100} 2^{i+1} - 2^i \Rightarrow \sum_{i=1}^{101} 2^i - 2^{i-1} \Rightarrow 2^{101} - 1$$

$$\text{c) } \sum_{i=1}^{100} \left(\frac{1}{i} - \frac{1}{i+1} \right) = - \sum_{i=1}^{100} \frac{1}{i+1} + \frac{1}{i} = \left(\frac{1}{101} - 1 \right) =$$

$$\frac{101}{101} - \frac{1}{101} = \frac{100}{101}$$

$$\textcircled{6} \text{ a) } \sum_{i=1}^N c a_i = c \sum_{i=1}^N a_i$$

$$\sum_{i=1}^N c a_i = c a_1 + c a_2 + \dots + c a_N = c(a_1 + a_2 + \dots + a_N) =$$

$$c \sum_{i=1}^N a_i$$

$$\sum_{j=1}^1 a_j + \sum_{j=1}^2 a_j \quad \text{etc.}$$

$\downarrow$ $\downarrow$
 $i=1$ $i=2$

Pg 280-281

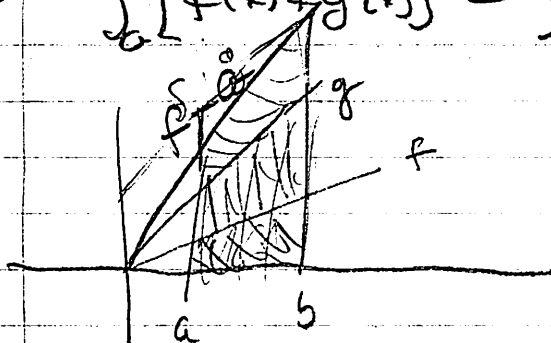
66 $\sum_{i=1}^4 \left(\sum_{j=1}^i a_j \right) = a_1 + (a_1 + a_2) + (a_1 + a_2 + a_3) + (a_1 + a_2 + a_3 + a_4)$

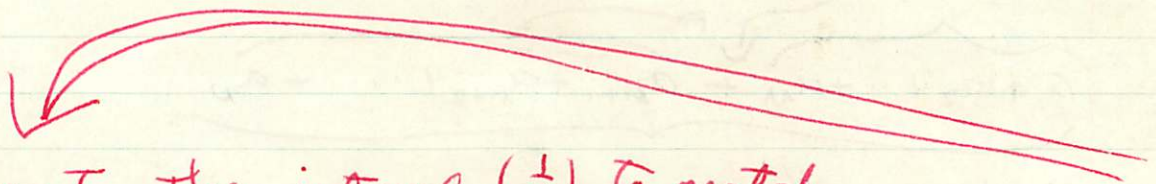
c) $\sum_{i=1}^m a_i + \sum_{i=m+1}^N a_i = \sum_{i=1}^N a_i$

$\underbrace{a_1 + a_2 + \dots + a_m + a_{m+1} + a_{m+2} + \dots + a_N}_{\sum_{i=1}^N a_i}$

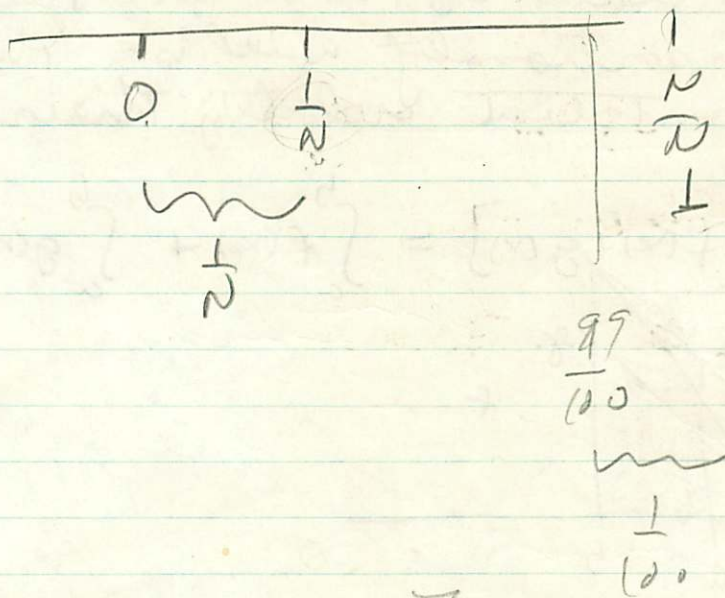
7. The distance traveled by the object traveling at a rate of $Cf(x)$ will be proportional to the distance of the other object. The constant of proportionality will be the same as the constant relating their speeds.

⑧ $\int_a^b [f(x) + g(x)] = \int_a^b f(x) + \int_a^b g(x)$





we want the interval $(\frac{1}{2})$ to match
up with a sample point in $f(\frac{1}{2})$ as
it counts up.



$$\left[\frac{x_i - x_{i-1}}{2} \right]$$

$$\textcircled{9} \sum_{i=1}^4 (-x)^i = -x - x^2 - x^3 - x^4$$

a)

$$\textcircled{9} \sum_{k=1}^4 \frac{x^k}{k} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$$

$$\textcircled{9} \sum_{k=1}^4 \frac{(-x)^k}{k^2} = -x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4}$$

$$\textcircled{10} \sum_{i=1}^n f\left(\frac{i}{n}\right) \left(\frac{1}{n}\right) \approx \int_0^1 f(x) dx$$

$\uparrow$ PTab Sampling
 $\uparrow$ interval length

The reason for limits $0 \rightarrow 1$ is that sample PT has $\frac{1}{n}$ rather than $i + \frac{1}{n}$ for example see next pg.

Sum of the Pts $\therefore$ by def this is approx of $\int_0^1 f(x) dx$

b) $\frac{1}{n}$ c) $\frac{1}{n}$ mesh is interval length (longest)

d) Sample pt RT hand endpt.

$$\textcircled{11} \frac{1}{100} \sum_{i=1}^{100} f\left(\frac{i}{100}\right) \approx \int_0^1 f(x) dx$$

$\uparrow$ mesh $\uparrow$ samplept
 Sum by def.

$$\textcircled{12} \text{ a) } \sum_{i=1}^{200} \left(\frac{i}{100}\right)^3 \frac{1}{100} \approx \int_0^2 x^3 dx$$

Notice there are 200 intervals each of length $\frac{1}{100}$. interval length 2,

$$\text{b) } \sum_{i=1}^{100} \left(\frac{i-1}{100}\right)^4 \frac{1}{100} \approx \int_0^1 x^4 dx$$

$$\text{c) } \sum_{i=101}^{300} \left(\frac{i}{100}\right)^5 \frac{1}{100} \approx \int_1^3 x^5 dx$$

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$$\frac{1}{N} = \frac{\text{Interval length}}{\text{# of partitions}} = \text{width}$$

$$(13) \quad \frac{1}{N^3} \sum_{i=1}^N i^2 = \sum_{i=1}^N \frac{1}{N^3} i^2 = \sum_{i=1}^N \left(\frac{i}{N}\right)^2 \frac{1}{N} \approx \int_0^1 x^2 dx$$

we need to regroup so we can have the interval length match up with the sample point. The $\left(\frac{i}{N}\right)^2$ is $f\left(\frac{i}{N}\right)$ so f must be x^2 .

$$b) \quad \frac{1}{4^3} \sum_{i=1}^4 i^2 = \frac{1}{64} (1+4+9+16) = \frac{30}{64} = \frac{15}{32}$$

$$c) \quad \int_0^1 x^2 dx = \left. \frac{x^3}{3} \right|_{x=0}^1 = \frac{1}{3}$$

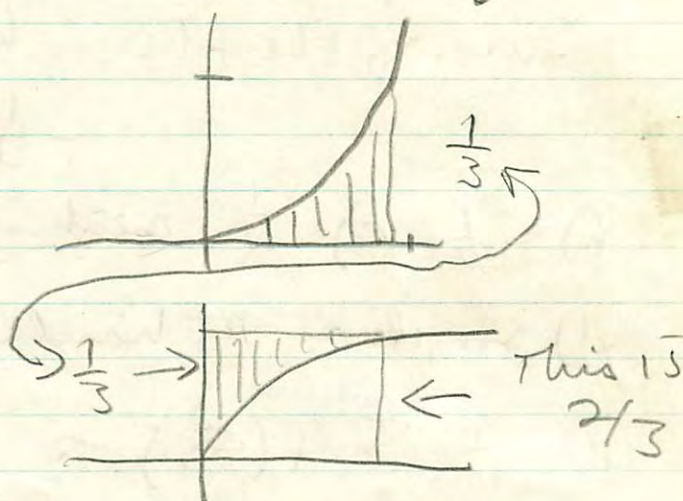
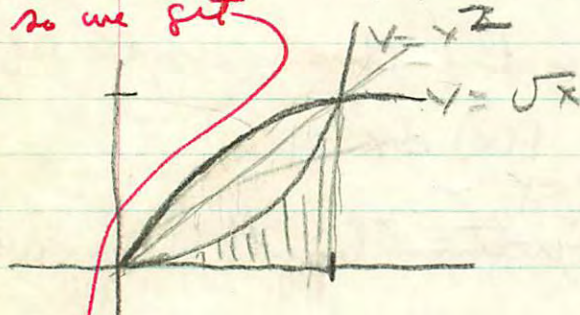
P61m 16e
Pg 264

$$d) \quad \lim_{N \rightarrow \infty} \frac{1}{N^3} \sum_{i=1}^N i^2 = \frac{1}{3} \quad \text{because as } N \rightarrow \infty \text{ the}$$

$\frac{1}{N^3} (1^2 + 2^2 + 3^2 + \dots + N^2)$ but second factor has a sum of $\frac{N(N+1)(2N+1)}{6}$ approximation approaches $\int_0^1 x^2 dx = \frac{1}{3}$.

so we get

(14)



(15)

$$3 \leq f(x) \leq 5$$

$$\int_0^4 f(x)$$

Both answers,

Largest 20 5 everywhere
Smallest 12 3 everywhere

$$\frac{1}{N^3} \left(\frac{N(N+1)(2N+1)}{6} \right) = \frac{1}{N^3} \left(\frac{2N^3 + 3N^2 + N}{6} \right) = \frac{2 + \frac{3}{N} + \frac{1}{N^2}}{6}$$

$$\lim_{N \rightarrow \infty} \frac{1}{6} \left(2 + \frac{3}{N} + \frac{1}{N^2} \right) = \frac{2}{6} = \frac{1}{3}$$

Summary of $\Sigma \rightarrow \int \frac{1}{N} \rightarrow \Sigma$

$$\sum_{i=1}^N \overbrace{f(\bar{x}_i)}^{\text{mesh}} (x_i - x_{i-1})$$

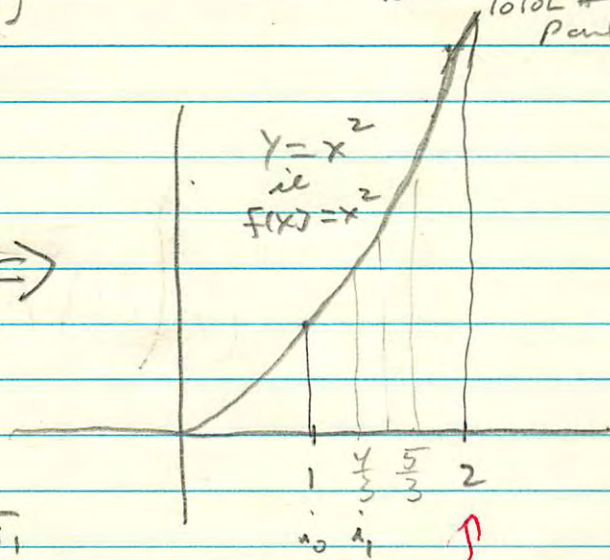
$$\text{mesh} \Rightarrow \frac{1}{N} = \frac{\text{length of } [a, b]}{\text{Total \# of Partitions}}$$

$N - 1 + 1 =$

N is # of Partitions

$$\sum_{i=1}^N \left(\frac{N+i}{N} \right)^2 \left(\frac{1}{N} \right) \Rightarrow$$

$i=1$ gives 1st sample pt.
 $i=N$ gives LAST sample pt.



Divide into 3 intervals so $\bar{x}_1 = \frac{4}{3}$

$$\frac{a+b}{3} = \frac{4}{3}$$

$$\frac{a+nb}{3} = 2$$

$$a+b=4$$

$$a+nb=6$$

$$a+b=4$$

$$(N-1)b=2 \quad N=3 \text{ so } b=1 \text{ so } a=3$$

for N Partitions:

$$\left(\frac{N+i}{N} \right)$$

↑

skiz.

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(16)

$$c > d > 0$$

$$c^3 > cd\left(\frac{c+d}{2}\right) > d^3$$

$$c > d > 0$$

$$c+c > d+d > d$$

$$c+c > c+d > c$$

$$\frac{c+d}{2} > d > \frac{d}{2}$$

$$c > \frac{c+d}{2} > \frac{c}{2}$$

$$c > d > 0$$

$$c^3 > d^3 > 0$$

$$a) \left(c^3 > cd\left(\frac{c+d}{2}\right) > d^3 \right) \frac{c-d}{c^3 d^3}$$

$$b) \rightarrow \frac{c-d}{d^3} > \frac{c^2-d^2}{2cd^3} > \frac{c-d}{c^3}$$

$$\rightarrow \frac{1}{2d^2} - \frac{1}{2c^2} >$$

$$c) b > a > 0$$

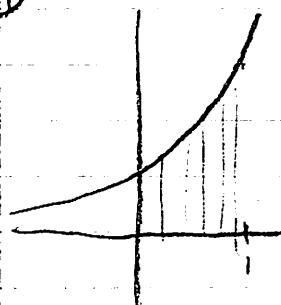
$$\int_a^b \frac{1}{x^3} dx = \frac{1}{2a^2} - \frac{1}{2b^2}$$

$$\frac{c-d}{c^3} < \frac{1}{2d^2} - \frac{1}{2c^2} < \frac{c-d}{d^3}$$

(17) area under curve is discretized, since area is found by summing rectangles as is total distance by summing short distances. Both Done the same way.

- (18) a) choose sample #s which are not dyadic.
 b) choose only dyadic #s.
 c) approx sums do not approach a single limit. No integral.

(19)

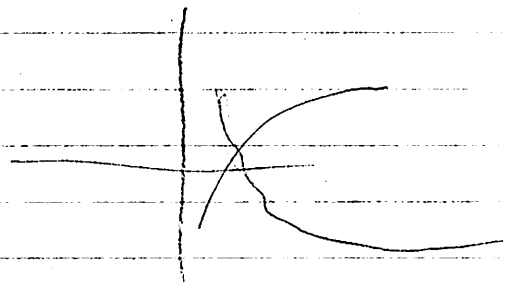


$$\sum_{i=1}^N$$

$$i+1 \left(\frac{i}{1+i} \right)$$

$$F(x) = \ln f(x) \quad G(x) = \ln g(x)$$

$$F''(x) > 0$$



Test ideas
Solve for b

$$\int_0^b \cos x \, dx = \frac{1}{2}$$

$$\sin b = \frac{1}{2}$$

$$b = \sin^{-1} \frac{1}{2}$$

$$\therefore b = \frac{\pi}{6} \text{ must be in radians.}$$

1st Fundamental Thm of Calculus.

If f is continuous on $[a, b]$ and if f is the derivative of F ($f = F'$) then

$$\int_a^b f(x) dx = F(b) - F(a)$$

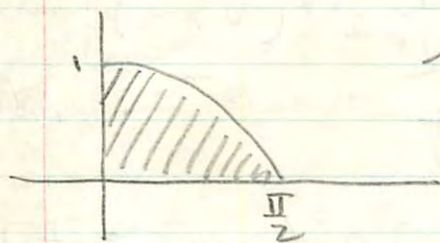
for example we know $\int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{3}$

from past experience

if $f(x) = x^2$ then $F(x) = \frac{x^3}{3}$

and $\int_a^b x^2 dx = F(b) - F(a) = \frac{b^3}{3} - \frac{a^3}{3}$

eg I consider the area bounded by x-axis
y axis and $f(x) = \cos x$.



if Area = $\int_0^{\pi/2} \cos x dx$.

we know if $f = \cos x$ then
 $F(x) = \sin x$ so

$$\int_0^{\pi/2} \cos x dx = \sin \frac{\pi}{2} - \sin 0 = 1 - 0 = 1$$

there is exactly 1 sq unit of area so
bounded.



So: In order to use the 1st Fundamental Thm
of Calculus you must be able to find a
fctn whose derivative you know.

lecture cont

Pg 283-287

eg 2 what if a, b are neg? this had effect in chart 7, however the 1st fundamental thm of calculus only states $a < b$. thus whether a, b are negative or not has NO effect.

eg 3

$$\int_0^1 e^{\sqrt{x}} dx$$

we need $F(x)$ $f(x) = e^{\sqrt{x}}$

$$\frac{1}{2\sqrt{x}} e^{\sqrt{x}}$$

we are trying to end up w/ $F' = f$, so we try a few to see what happens

$$2\sqrt{x} e^{\sqrt{x}}$$

$$2\sqrt{x} \cdot \frac{1}{2\sqrt{x}} e^{\sqrt{x}} = e^{\sqrt{x}}$$

$$+ \frac{1}{\sqrt{x}} e^{\sqrt{x}}$$

Extra need to fit w/ok this

$$x e^{\sqrt{x}} = e^{\sqrt{x}} + \frac{x}{2\sqrt{x}} e^{\sqrt{x}}$$

it is 2 times

$$x e^{\sqrt{x}} - e^{\sqrt{x}} = e^{\sqrt{x}} + \frac{x}{2\sqrt{x}} e^{\sqrt{x}} - \frac{1}{2\sqrt{x}} e^{\sqrt{x}}$$

so we subtract out $2 e^{\sqrt{x}}$

$$2\sqrt{x} e^{\sqrt{x}} - 2 e^{\sqrt{x}}$$

$$e^{\sqrt{x}} + \frac{1}{\sqrt{x}} e^{\sqrt{x}} - \frac{1}{\sqrt{x}} e^{\sqrt{x}}$$

$$\text{so } (2\sqrt{x} e^{\sqrt{x}} - 2 e^{\sqrt{x}})' = [2e^{\sqrt{x}}(\sqrt{x}-1)]' = e^{\sqrt{x}}$$

finally we are ready to apply the 1st fund. thm of calc,

$$\int_0^1 e^{\sqrt{x}} dx = 2e'(1-1) - 2e^0(0-1) = +2$$

note: this thm works nicely if you can find F . finding F is not always this easy.

What do we know about e^x ?
 $d(?) = e^x$ means? implies e^x
 so... what if I were e^x ?

1st we guess at a F . Since F is an e.fctn, we know F' is also an e.fctn.
~~1st we look at F 's derivative.~~

It looks like we need to get rid of the $2\sqrt{x}$ in the denom. so we multiply by this and take the derivative again.

$$e^{\sqrt{x}}$$

$$e^u \frac{du}{dx}$$

$$\frac{1}{2\sqrt{x}} e^{\sqrt{x}}$$

$$2\sqrt{x} e^{\sqrt{x}}$$

$$\frac{e^{\sqrt{x}}}{\sqrt{x}} + \frac{2\sqrt{x}}{2\sqrt{x}} e^{\sqrt{x}} = \frac{e^{\sqrt{x}}}{\sqrt{x}} + e^{\sqrt{x}}$$

Now we have an extra part: $\frac{e^{\sqrt{x}}}{\sqrt{x}}$ we need to get

rid of. Remember our 1st step we had $\frac{e^{\sqrt{x}}}{2\sqrt{x}}$. This almost

the same as an extra part. So make another guess and take the derivative.

$$2\sqrt{x} e^{\sqrt{x}} - 2e^{\sqrt{x}}$$

$$\frac{e^{\sqrt{x}}}{\sqrt{x}} + e^{\sqrt{x}} - \frac{e^{\sqrt{x}}}{\sqrt{x}} = e^{\sqrt{x}} \quad \text{So we have found}$$

$$F = 2\sqrt{x} e^{\sqrt{x}} - 2e^{\sqrt{x}}$$

$$1) a) e^{x^2} \Rightarrow 2x e^{x^2}$$

$$b) \frac{x^3}{3} \Rightarrow x^2$$

$$c) \frac{1}{2} \sin^{-1} 2x \Rightarrow \frac{1}{\sqrt{1-4x^2}}$$

$$d) -\frac{1}{2x^2} \Rightarrow \frac{4x}{4x^4} = \frac{1}{x^3}$$

$$e) \frac{1}{2} \ln\left(\frac{1-x}{1+x}\right) = \frac{1}{2} [\ln(1-x) - \ln(1+x)]$$

$$\frac{1}{2} \left[\frac{-1}{1-x} - \frac{1}{1+x} \right] = \frac{1}{2} \left[\frac{-1-x-1+x}{1-x^2} \right]$$

$$\frac{-2}{2(1-x^2)} = \frac{-1}{1-x^2} = \frac{1}{x^2-1}$$

$$f) \frac{1}{2} \cos^{-1} 2x \Rightarrow \frac{2}{2 \cdot 1+4x^2} = \frac{1}{1+4x^2}$$

$$g) x \sin x + \cos x$$

$$\sin x + x \cos x - \sin x = x \cos x.$$

$$h) -\cos x \Rightarrow \sin x$$

$$i) -e^{-x} \Rightarrow e^{-x}$$

$$j) \ln(\sec x + \tan x) \Rightarrow \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$

$$\frac{\sec x (\tan x + \sec x)}{(\sec x + \tan x)} = \sec x = \frac{1}{\cos x}$$

$$k) \sqrt{1+2x} \Rightarrow \frac{2}{2\sqrt{1+2x}} = \frac{1}{\sqrt{1+2x}}$$

$$l) \ln x \Rightarrow \frac{1}{x}$$

$$(2) \quad y = e^{-x} \quad \int_0^2 e^{-x} dx = -e^{-2} - (-e^0) = -e^{-2} + 1$$

$$(3) \quad \int_0^1 \frac{1}{1+x^2} dx = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}$$

$$(4) \quad \int_1^4 \frac{1}{t} dt = \ln 4 - \ln 1 = \ln 4$$

$$(5) \quad \int_0^2 2x e^{x^2} dx = e^{2^2} - e^{0^2} = e^4 - 1$$

$$(6) \quad \int_0^\pi \sin x = -\cos \pi + \cos 0 = 1 + 1 = 2$$

$$(7) \quad \int_2^3 \frac{1}{x^3} dx = \frac{-1}{2 \cdot 3^2} + \frac{1}{2 \cdot 2^2} = -\frac{1}{18} + \frac{1}{8}$$

$$= \frac{-4}{2^3 \cdot 3^2} + \frac{9}{2^3 \cdot 3^2} = \frac{5}{72}$$

$$(8) \quad \int_0^{\frac{\pi}{3}} \frac{1}{\cos x} dx = \ln(\sec \frac{\pi}{3} + \tan \frac{\pi}{3}) - \ln(\sec 0 + \tan 0) = \ln(2 + \sqrt{3})$$

$$(9) \quad \int_0^{\frac{1}{2}} \frac{1}{1-x^2} dx = -\frac{1}{2} \ln \left(\frac{1-\frac{1}{2}}{1+\frac{1}{2}} \right) = -\frac{1}{2} \ln \left(\frac{\frac{1}{2}}{\frac{3}{2}} \right) = -\frac{1}{2} \ln \left(\frac{1}{3} \right) = \frac{1}{2} \ln 3$$

$$(10) \quad \int_0^{\frac{\pi}{2}} t \cos t dt \Rightarrow F = t \sin t + \cos t$$

$$= \frac{\pi}{2} \sin \frac{\pi}{2} + \cos \frac{\pi}{2} - 0 \sin 0 - \cos 0 = \frac{\pi}{2} + 0 - 0 - 1 = \frac{\pi}{2} - 1$$

Ans in Book wrong.

Pg 287

$$(11) \int_0^4 \frac{1}{\sqrt{1+2x}} dx \Rightarrow F = (1+2x)^{\frac{1}{2}} \\ = (1+8)^{\frac{1}{2}} - (1+0)^{\frac{1}{2}} = 3 - 1 = 2$$

$$(12) \int_0^{\frac{1}{4}} \frac{1}{\sqrt{1-4x^2}} dx \Rightarrow F = \left(\frac{1}{2}\right) \sin^{-1} 2x \\ \frac{1}{2} \sin^{-1} \frac{1}{2} - \frac{1}{2} \sin^{-1} 0 \\ \frac{1}{2} \cdot \frac{\pi}{6} = \frac{\pi}{12}$$

$$(13) 5 + \sin x = f(x)$$

a) $f'(x) = \cos x$ if $f(x) = \sin x$, $g(x) = \sin x + K$
 $f' = g'$ because fctns differ only in a constant.

b) Same answer because the constant is subtracted in the 2nd portion of the definite integral.

$$(14) \int_0^1 e^{-x^2} dx \quad x_0 = 0 \quad x_1 = \frac{1}{3} \quad x_2 = \frac{2}{3} \quad x_3 = 1 \\ \bar{X}_1 = .3 \quad \bar{X}_2 = .4 \quad \bar{X}_3 = .8$$

$$e^{-(.3)^2} \left(\frac{1}{3} - 0\right) + e^{-(.4)^2} \left(\frac{2}{3} - \frac{1}{3}\right) + e^{-(.8)^2} \left(1 - \frac{2}{3}\right)$$

$$\frac{1}{3} (e^{-.09} + e^{-.16} + e^{-.64}) = \frac{1}{3} (2.29) = .764$$

this pblm is important since it reminds everyone that we have built upon the idea of sampling pts in the intervals. this idea will show up again in the Proofs of the FTC Proofs in 8.3

(15) a) $\int_0^{\frac{\pi}{2}} \sin 2x \, dx \Rightarrow F = -\frac{1}{2} \cos 2x$

$$-\frac{1}{2} \cos \pi + \frac{1}{2} \cos 0 = \frac{1}{2} + \frac{1}{2} = 1$$

b) $\int_0^{\pi} \sin x^2 dx$ cannot be done using 1st Fund. Thm

c) $\int_0^1 \frac{x^2 dx}{e^{x^3}} \Rightarrow F = \frac{1}{3} \ln e^{x^3} = \frac{1}{3} x^3$

~~$\frac{1}{3} \ln$~~ $\frac{1}{3} i^3 - \frac{1}{3} \cdot 0^3 = \frac{1}{3}$

$$\frac{3x^2 e^{x^3}}{3e^{x^3}} \quad \frac{x^2}{e^{x^3}}$$

$$\ln \frac{1}{e^{x^3}} \quad x^2 e^{-x^3}$$

$$F = \frac{-1}{3} e^{-x^3} \Rightarrow \left(\frac{1}{3} x^2 e^{-x^3} = \frac{x^2}{e^{x^3}} \right)$$

$$-\frac{1}{3e^{13}} - \frac{10}{3e^0} = -\frac{1}{3e} + \frac{1}{3}$$

16 4) $\int_0^1 x^2 e^{x^3} dx$ $F = \frac{1}{3} e^{x^3}$ $\frac{1}{3} e^1 - \frac{1}{3} e^0 = \frac{1}{3}(e-1)$

B) $\int_0^1 e^{x^3} dx$ cannot be done w/ FT of calc

this is c) $\int_0^{\frac{\pi}{2}} x \sin x^2 dx$ $-\frac{1}{2} \cos x^2 = F$
 nearly

the answer,
given as $\frac{1}{2} \cos 0 - \frac{1}{2} \cos\left(\frac{\pi}{2}\right)^2 = -\frac{1}{2} \left(1 - \cos\left(\frac{\pi}{2}\right)^2\right)$

15c in answers.

17 a) $\int_0^1 \sqrt[3]{x+x^2} dx$ cannot be done w find the abc

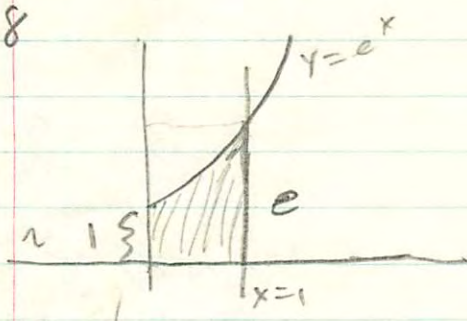
b) $\int_0^1 \sqrt[3]{1+x} dx$ $\frac{4}{3}(1+x)^{4/3} = F$

$$\frac{4}{3}(1+1)^{4/3} - \frac{4}{3}(1+0)^{4/3} = \frac{4}{3}(2^{4/3} - 1)$$

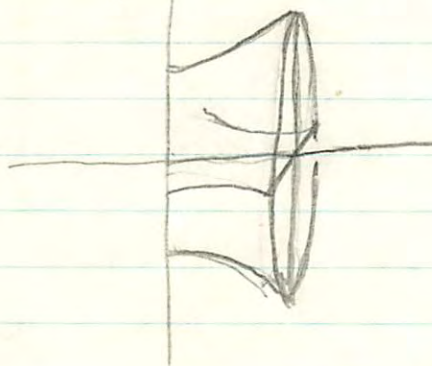
c) $\int_0^1 \frac{1}{\sqrt[3]{1+x}} dx$ $\frac{2}{3}(1+x)^{2/3} = F$

$$\frac{2}{3}(2)^{2/3} - \frac{2}{3}$$

18



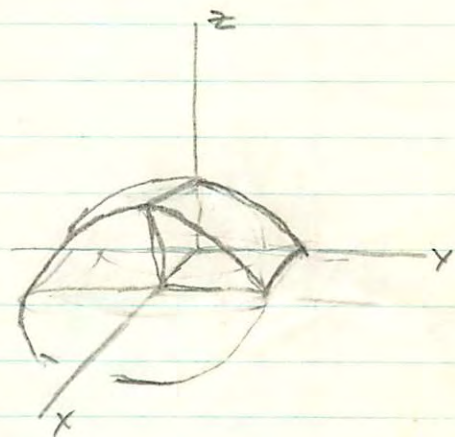
Area of Circle πr^2 but $r = e^x$



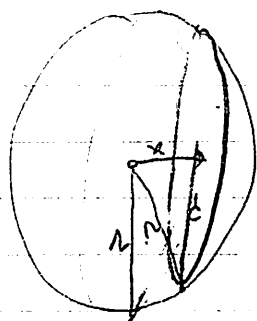
$$\int_0^1 \pi e^{2x} dx = \pi \int_0^1 e^{2x} dx$$

$$F = \frac{1}{2} e^{2x}$$

$\frac{\pi}{2}(e^2 - 1) = \text{area of required shape}$



19



$$a) d = \sqrt{r^2 - x^2}$$

$$b) A = \pi r^2 - \pi(r^2 - x^2)$$

$$2 \int_{x=0}^r \pi(r^2 - x^2) dx = 2 \int_0^r \pi r^2 dx - 2 \int_0^r \pi x^2 dx$$

$$2 \left[\pi r^2 x \right]_0^r - 2 \left[\frac{\pi x^3}{3} \right]_0^r$$

$$2 \pi r^3 - \frac{2 \pi r^3}{3} = \frac{4 \pi r^3}{3}$$

20) a) $\int_0^\pi \sqrt{\sin x} dx$

b) $\int_0^\pi \pi \sin x dx$

c) $\int_0^\pi \pi \sin x dx \quad F = -\pi \cos x$

$$-\pi \cos \pi + \pi \cos 0 = \pi + \pi = 2\pi$$

This can be found using the fundamental theorem and "a" above correct,

Use t as the horizontal axis, t for temporary.

2nd FTC

let f be the height under a curve. (1 value)

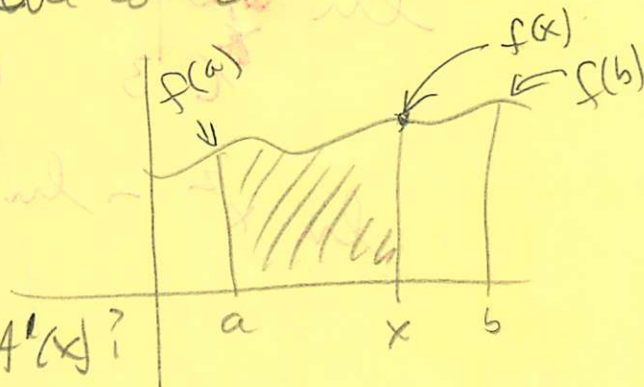
So, area under curve is

given by $A(t) = \int_{T=a}^{T=b} f(t) dt$



Now, suppose "a" is held fixed but the value "b" is allowed to change. We will call it "x".

So we have $A(x) = \int_a^x f(t) dt$



Now, how would we define $A'(x)$?

The der of mass is density; The der of distance is velocity and the der of area is height. so in this case $A'(x) = \text{height } f(t) \text{ evaluated at } t=x$ i.e. $A'(x) = f(x)$ thus.

$$\frac{d}{dx} \left[\int_a^x f(t) dt \right] = f(x)$$

The 2nd Fundamental Theorem of Calc.

chain rule version

$$\frac{d}{dx} \int_a^{g(x)} f(t) dt = f(g(x)) \cdot g'(x)$$



$$\left(\int_3^{x^2} \frac{1}{t} dt \right)' \xrightarrow{\text{1st FTC}} (\ln x^2 - \ln 3)'$$

or
↓ 2nd FTC

$$\frac{2x}{x^2} - 0$$

$$\frac{2}{x}$$

$$\frac{1}{x^2} \cdot 2x$$

$$\frac{2}{x}$$

$$\left(\int_a^{\star} f(t) dt \right)' \stackrel{\text{2nd FTC}}{=} f(\star) d\star$$

Given $g(x) = \int_3^{3x} \frac{1}{t} dt$

and $1 \leq t \leq 15$

So D_g $1 \leq 3x \leq 15$

$$\frac{1}{3} \leq x \leq 5$$

Then
The domain of g is ↗

$$\cancel{f(t) = t^2}$$

Let $g(x) = \int_{-3}^{3x} f(t) dt$ and $-6 \leq t \leq 3$

notice that the (-3) is in the interval $\uparrow$

Find Domain of $g(x)$.

~~Find $g(x)$~~

$$-6 \leq 3x \leq 3$$

$$-2 \leq x \leq 1$$

$$D_g \quad -2 \leq x \leq 1$$

$$g'(x) = f(3x) \cdot 3$$

Let's say
 $\rightarrow$ ~~pf~~ $f(t) = t^2$

Then $g(x) = \int_{-3}^{3x} t^2 dt$ what would $g'(x)$ be?

$$g'(x) = (3x)^2 \cdot 3$$

$$27x$$

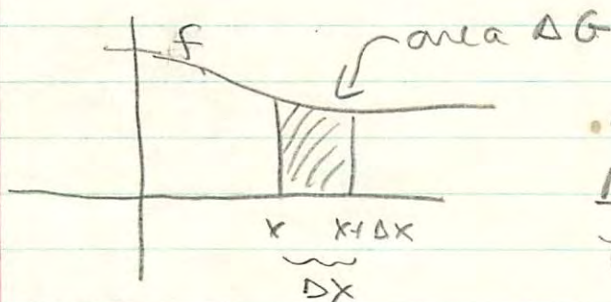
Briefly stated the 2nd FTC says

$$\frac{d\left(\int_a^x f(t) dt\right)}{dx} = f(x) \left(\frac{d\int_a^u f(t) dt}{dx} \right) = f(u) \frac{du}{dx}$$

where u is fctn of x

In other words, the derivative "undoes" the integral.

using the area version



as $\Delta x \rightarrow 0$ area ΔG actually approaches the value of the fctn f at x .

all we require at this stage is reasonable acceptance. Both fundamental thms will be proved in the next section.

Pg 291

Do 1-10 in class. Notice pblm 4, 5 have same answer even though different beginning pts. why? Because we are actually taking derivative of the constant's drop off. Consider using 1st FTC.

$$\int_3^x t^5 dx = \frac{x^6}{6} - \frac{3^6}{6} \quad \text{now take deriv,}$$

$$\frac{d\left(\frac{x^6}{6} - \frac{3^6}{6}\right)}{dx} = x^5$$

more ↓

Notice the corollary at top of pg 291. This will be useful later.

"Let f be continuous on an interval $[a, b]$, then f is the derivative of some F on $[a, b]$."

$$(1) \int_0^x \sqrt{1+t^3} dt$$

$$(2) \int_0^x \sqrt[3]{1+t^2} dt$$

$$(3) \int_0^x \tan^{-1} u^2 dt$$

$$(4) G(x) = \int_3^x t^5 dt \quad G'(x) = x^5 \quad \left. \begin{array}{l} \text{notice it} \\ \text{makes NO} \end{array} \right\}$$

$$(5) G(x) = \int_4^x t^5 dt \quad G'(x) = x^5 \quad \left. \begin{array}{l} \text{difference where} \\ \text{we START! compare} \end{array} \right\}$$

$$(6) \text{ let } y = x^5 + 3 \text{ and } g = x^5 - 3 \quad f' = g'$$

$$G(x) = \int_4^x \sec^3 t dt \quad G'(x) = \sec^3 x$$

$$(7) G(x) = \int_1^x \ln t dt \quad G'(x) = \ln x$$

$$(8) G(x) = \int_0^x \sqrt{t} dt \Rightarrow G'(x) = \sqrt{x}$$

$$(9) G(x) = \int_3^x e^{t^3} dt \quad G'(x) = e^{x^3}$$

$$(10) G(x) = \int_1^{x^3} \frac{1}{t} dt \quad \text{let } u = x^3 \quad \frac{du}{dx} = 3x^2$$

$$G(u) = \int_1^u \frac{1}{t} dt \quad G'(u) = \frac{1}{u} = \frac{dG}{du}$$

$$\frac{dG}{dx} = \frac{dG}{du} \cdot \frac{du}{dx} \Rightarrow \frac{1}{x^3} \cdot 3x^2 = \frac{3}{x}$$

$$(11) \quad G(x) = \int_1^{\sqrt{x}} \sqrt{1-t^2} \, dt \quad \text{let } u = \sqrt{x}$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$G(u) = \int_1^u \sqrt{1-t^2} \, dt \quad \therefore G'(u) = \sqrt{1-u^2}$$

$$G'(x) = \sqrt{1-x} \cdot \frac{1}{2\sqrt{x}} = \frac{\sqrt{1-x}}{2\sqrt{x}}$$

$$(13) \quad G(x) = \int_0^{-x} (3t^2 + 4t) \, dt \quad x < 0$$

$$\text{let } u = -x \quad G(u) = \int_0^u (3t^2 + 4t) \, dt$$

$$\frac{du}{dx} = -1$$

$$G'(u) = 3u^2 + 4u$$

$$G'(x) = [3(-x)^2 + 4(-x)] \cdot (-1) = -3x^2 + 4x$$

$$(12) \quad G(x) = \int_0^{\frac{1}{x}} e^{t^2} \, dt$$

$$\text{let } u = \frac{1}{x} \quad G'(u) = e^{u^2}$$

$$\frac{du}{dx} = -\frac{1}{x^2}$$

$$G'(x) = e^{\frac{1}{x^2}} \cdot \left(-\frac{1}{x^2}\right) = -\frac{e^{\frac{1}{x^2}}}{x^2}$$

$$(14) \quad G(x) = \int_3^{\sin x} \sqrt{1+t^2} \, dt$$

$$\text{let } u = \sin x$$

$$\frac{du}{dx} = \cos x$$

$$G'(u) = \sqrt{1+u^2}$$

$$G'(x) = (\sqrt{1+\sin^2 x}) \cos x$$

Notice from 3 to $\sin x$
this appears to be backwards,
wrt other probs.

Needs more work.

for related ex. sm. prob. use: $G(x) = \int_0^{\sin x} \sqrt{1-t^2} \, dt \Rightarrow$
(when completely simplified:

$$(\sqrt{1-\sin^2 x}) \cos x = \cos^2 x$$

$$(14) \quad G(x) = \int_3^{\sin x} \sqrt{1+t^2} dt \quad \text{let } u = \sin x$$

$$\frac{du}{dx} = \cos x$$

$$\frac{d(G(x))}{dx} = G'(x) = G'(u) \cdot u'(x) \quad \text{!}$$

$$\frac{d(G)}{du} \cdot \frac{du}{dx}$$

$$d(G(u)) = d\left(\int_3^u \sqrt{1+t^2} dt\right) \Rightarrow \left(\sqrt{1+u^2}\right) \cos x$$

$$\left(\sqrt{1+\sin^2 x}\right) \cos x$$

2nd version, using definition of derivative

$$G(u) = \int_3^u \sqrt{1+t^2} dt, \quad G'(u) = \lim_{h \rightarrow 0} \frac{G(u+h) - G(u)}{h}$$

$$\lim_{h \rightarrow 0} \frac{1}{h} \left[\int_3^{u+h} \sqrt{1+t^2} dt - \int_3^u \sqrt{1+t^2} dt \right] =$$

$$\lim_{h \rightarrow 0} \frac{1}{h} \left[\int_3^{u+h} \sqrt{1+t^2} dt + \int_u^3 \sqrt{1+t^2} dt \right] =$$

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_u^{u+h} \sqrt{1+t^2} dt$$

?

$$\lim_{h \rightarrow 0} \frac{1}{h}$$

?

$$(5) \quad G(x) = \int_1^{x^2} f(t) dt$$

$$\text{let } u = x^2 \quad G(u) = \int_1^u f(t) dt$$

$$\frac{du}{dx} = 2x$$

$$\frac{G'(u)}{dt} = f(u) \frac{du}{dx}$$

$$G'(x) = 2x f(x^2).$$

$$\text{let } f = x^3$$

$$(16) \quad \lim_{\Delta x \rightarrow 0} \frac{\int_2^{3+\Delta x} x^3 dx - \int_2^3 x^3 dx}{\Delta x}$$

def of der
for $f'(3)$

$$d(G(x)) = \left(\int_2^u x^3 dx \right) \Rightarrow \left[\frac{u^4}{4} \right]_{u=2}^{u=3} = 27$$

or

$$(17) \lim_{x_2 \rightarrow x_1} \frac{\int_0^{x_2} e^{t^2} dt - \int_0^{x_1} e^{t^2} dt}{x_2 - x_1}$$

let $f(x) = e^{x^2}$ so $d(G(x)) = e^{x^2} \Big|_{x=x_1} = e^{x_1^2}$

$$(18) \lim_{h \rightarrow 0} \frac{\int_1^{x+h} \sqrt{\sin t} dt - \int_1^x \sqrt{\sin t} dt}{h}$$

let $f(x) = \sqrt{\sin x}$

let $u = \sin x$ so $\frac{du}{dx} = \cos x$
 $d(G(x)) = \sqrt{\sin u} \Big|_{u=x} = \sqrt{\sin x}$

(17)

let $h = x_2 - x_1$
 then $x_2 = x_1 + h$

$$\frac{\int_0^{x_1+h} e^{t^2} dt - \int_0^{x_1} e^{t^2} dt}{h}$$

$\lim_{h \rightarrow 0}$

Lemma 1

$$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$$

1st part

+ 2nd part =

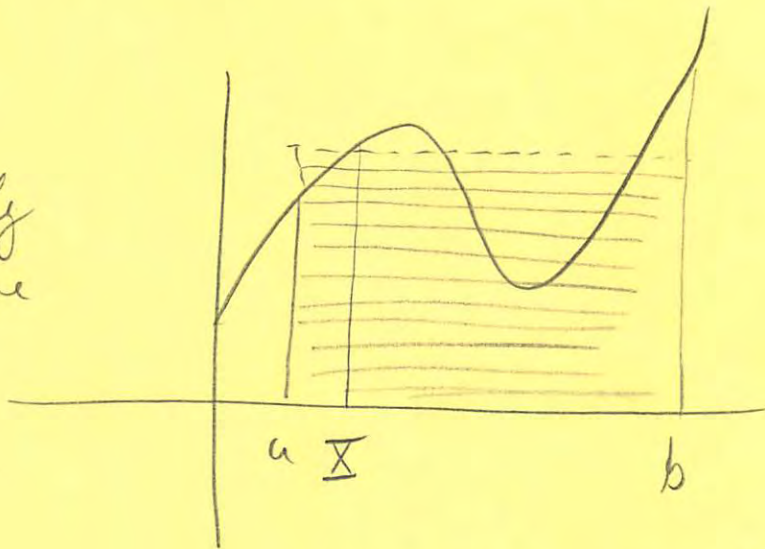
whole.



Lemma 2

$$\int_a^b f(x) dx = f(\xi) (b-a)$$

There exists a ξ in $[a, b]$
such that the area of the
rectangle $\xi(b-a)$ is exactly
equal to area under the curve
as described: $\int_a^b f(x) dx$.



consider $\frac{\text{Area}}{\text{Base length}} = \text{height}$

$$\frac{\int_a^b f(x) dx}{b-a} = \text{height} = f(\xi)$$

$$\text{so } \int_a^b f(x) dx = f(\xi) (b-a)$$

The actual proof would be
in the form of using
the under and over estimates
as sums and establishing that
 ξ is within the interval
 $[a, b]$ we already did this
in CHAPT 7.

2nd FTC

let f be continuous on $[a, b]$. let $G(x) = \int_a^x f(t) dt$
 for $a < x < b$ then G is differentiable and its
 derivative is f i.e. $G'(x) = f(x)$ #2

Define Δx — small change in x

ΔG — small change in G caused by Δx
 i.e. $G(x + \Delta x) - G(x) = \Delta G$

Given $G(x) = \int_a^x f(t) dt$

so $G(x + \Delta x) = \int_a^{x + \Delta x} f(t) dt$

$$G(x + \Delta x) = \int_a^x f(t) dt + \int_x^{x + \Delta x} f(t) dt$$

Lemma 1

$\Delta G = G(x + \Delta x) - G(x)$ becomes.

$$\Delta G = \int_a^x f(t) dt + \int_x^{x + \Delta x} f(t) dt - \int_a^x f(t) dt$$

$$\Delta G = \int_x^{x + \Delta x} f(t) dt$$

$$\Delta G = \int_x^{x + \Delta x} f(t) dt = f(\xi) [x + \Delta x - x]$$

Lemma 2

$$\Delta G = f(\xi) \Delta x \quad \div \Delta x$$

thus $f(\xi) = \frac{\Delta G}{\Delta x} \stackrel{\text{which is}}{=} \frac{G(x + \Delta x) - G(x)}{\Delta x}$

Now, as we take limit at $\Delta x \rightarrow 0$

#3

$$\lim_{\Delta x \rightarrow 0} f(\bar{x}) = \lim_{\Delta x \rightarrow 0} \frac{G(x+\Delta x) - G(x)}{\Delta x}$$

Since $\bar{x} \in [x, x+\Delta x]$
And when $\Delta x \rightarrow 0$
our interval becomes $\rightarrow$

$G'(x)$ 2nd def of Der.

$\bar{x} \in [x, x]$, then

$\bar{x}$ becomes $\dots x$

Thus

$$f(x) = G'(x)$$

End of 2nd FTC

Next consider 1st FTC.

If f is continuous on $[a, b]$ and if F is the

der of F' ; i.e. $(F')' = f$ Then

$$\int_a^b f(x) = F'(b) - F'(a)$$

let $G(x) = \int_a^x f(t) dt$ according to 2nd FTC $G'(x) = f(x)$

but we assumed here that $F'(x) = f(x)$ so we have two fct's with the same derivative. This means that F' and G can differ by at most a constant.

$$\text{So } G(x) = f'(x) + C$$

$$\text{So } G(b) = f'(b) + C$$

$$G(a) = f'(a) + C$$

← #4
Subtract this line from

$$G(b) - G(a) = f'(b) + C - (f'(a) + C)$$

we know $G(a)$
is zero because

$$G(x) = \int_a^x f(t) dt \text{ and}$$

$$G(a) = \int_a^a f(t) dt = 0$$

Also

we know $G(b) =$
 $\int_a^b f(t) dt.$

$$\int_a^b f(t) dt \stackrel{V}{=} f'(b) - f'(a) \quad \text{QED.}$$

We will prove the 2nd FTC and then use it to prove the 1st FTC.

Lecture
p. 292-298
Pg 1

FTC #2 Then: let f be continuous on $[a, b]$. let

$$G(x) = \int_a^x f(t) dt \quad \text{for } a < x < b \text{ then}$$

G is differentiable and that derivative is f .

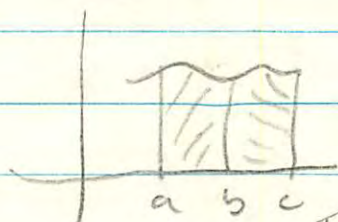
i.e. $G'(x) = f(x)$.

We need 3 lemmas; lemmas are used when their proof within a proof would be a distraction to the desired proof.

Lemma 1 if $a < b < c$ then

$$\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx$$

area of whole area of 1st part area of 2nd part



Lemma 2 used if $\int_a^b f(x) dx$ exists $\iff$ if $m \leq f(x) \leq M \forall x \in [a, b]$ then

Lemma 3
Lemma 2 and 3 are used in proofs.

Notice m, M are range values $m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$

(if $a > b$ then reverse so $a < b$)

Proof: $m \neq M$ are constants

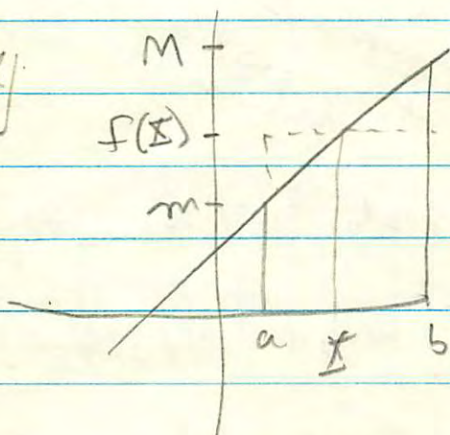
if $[a, b]$ is partitioned into N sub-interval (or sections) each of length $(x_i - x_{i-1})$ and ξ_i is the sample pt of each then

$$\int_a^b f(x) dx \text{ is approximated by } \sum_{i=1}^N f(\xi_i)(x_i - x_{i-1})$$

$$\frac{\int_a^b f(x) dx}{b-a} = \frac{\text{area under curve.}}{\text{interval length}}$$

height of
a rectangle
this rectangle
has same area
as that trapped
under $f(x)$ between
 a and b .

Graphically.



$(b-a) f(x)$ has same area
as under curve

Go to pg 212

Since $\sum_{i=1}^N (x_i - x_{i-1})$ is a telescoping sum, when we take the limit as the mesh goes to zero, this summation is equal to $b-a$.

Since $m \leq f(x)$ then

$$m \sum_{i=1}^N (x_i - x_{i-1}) \leq \sum_{i=1}^N f(x_i)(x_i - x_{i-1})$$

The case for M is set up in a like wise manner.

So this -

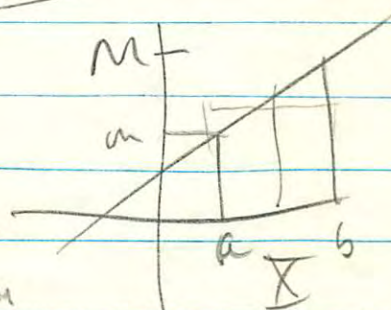
as mesh $\rightarrow 0$ $N \rightarrow \infty$
 $\sum (x_i - x_{i-1})$ still $= b-a$

$$\lim_{\text{mesh} \rightarrow 0} \left[m \sum_{i=1}^N (x_i - x_{i-1}) \leq \sum_{i=1}^N f(x_i)(x_i - x_{i-1}) \leq M \sum_{i=1}^N (x_i - x_{i-1}) \right]$$

becomes:

By Def lemma 2 $\rightarrow m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$
 Divide by $b-a$

$$m \leq \frac{\int_a^b f(x) dx}{b-a} \leq M$$



Now Notice that this $\frac{\int_a^b f(x) dx}{b-a}$ is a fixed number
 to Lemma 3 p 197 page

The intermediate value theorem says $\exists$ at least one $\xi \in [a, b] \Rightarrow f(\xi) = \frac{1}{b-a} \int_a^b f(x) dx$

thus we get

$$f(\xi)(b-a) = \int_a^b f(x) dx \quad \text{Q.E.D.}$$

Lemma 3

 $a < b$ M max m minmax then pg 166.
guarantees

$$m < f(x) < M \text{ when } a \leq x \leq b$$

From Lemma 2 we have

$$m < \frac{\int_a^b f(x) dx}{b-a} < M$$

$$f(x) \text{ not necessarily equal to } \frac{\int_a^b f(x) dx}{b-a}$$

But Intermediate Value theorem pg 94 says

that our x can be the one such that

$$f(x) = \frac{\int_a^b f(x) dx}{b-a}$$

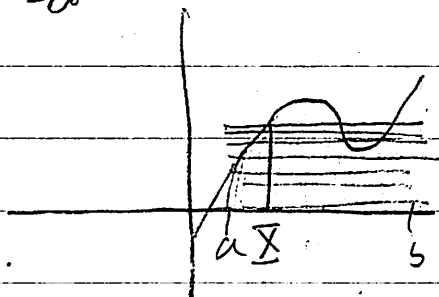
in since x is in the interval we choose
this x to be one of many possible choices
 such that $f(x) = \frac{\int_a^b f(x) dx}{b-a}$

$$\text{thus } (b-a) f(x) = \int_a^b f(x) dx \quad \text{Q.E.D. of Lemma 3}$$

Lemma 1
$$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$$

Now Lemma 2
$$\int_a^b f(x) dx = f(\bar{x})(b-a)$$

This is an approximation
simplification
but generally
what they
do in the book.



There exists a $\bar{x}$ in $[a, b]$
such that the height
 $f(\bar{x})$ times width $b - a$
= area under curve from
 $[a, b]$

$$a \leq f(x)$$

Proof of 2nd Fundamental theorem of Calculus

Let f be continuous on $[a, b]$, let $G(x) = \int_a^x f(t) dt$
 for $a < x < b$ then G is differentiable and its
 derivative is f i.e.

$$G'(x) = f(x)$$

Define Δx = small change in x

ΔG = small change in G caused by x changing
 by Δx i.e. $G(x + \Delta x) - G(x) = \Delta G$

we are given $G(x) = \int_a^x f(t) dt$

So
$$G(x + \Delta x) = \int_a^{x + \Delta x} f(t) dt$$

$$= \int_a^x f(t) dt + \int_x^{x + \Delta x} f(t) dt \quad (1^{st} \text{ lemma})$$

$\Delta G = G(x + \Delta x) - G(x)$ so

$$\Delta G = \int_a^x f(t) dt + \int_x^{x + \Delta x} f(t) dt - \int_a^x f(t) dt = \int_x^{x + \Delta x} f(t) dt$$

according to Lemma 3

$$\Delta G = \int_x^{x + \Delta x} f(t) dt = f(\xi)(x + \Delta x - x)$$

i.e. $\Delta G = f(\xi) \Delta x$

thus $\frac{\Delta G}{\Delta x} = f(\xi) = \frac{G(x + \Delta x) - G(x)}{\Delta x}$

Now take the limit as $\Delta x \rightarrow 0$

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta G}{\Delta x} = \lim_{\Delta x \rightarrow 0} f(\bar{x}) = \lim_{\Delta x \rightarrow 0} \frac{G(x+\Delta x) - G(x)}{\Delta x}$$

By Definition of Derivative

Since $\bar{x} \in [x, x+\Delta x]$

$G'(x)$

and when $\Delta x \rightarrow 0$ our

interval becomes $\bar{x} \in [x, x]$

so $\bar{x}$ must become simply x .

$F(x)$

Thus $F(x) = G'(x)$ QED

Proof of 1st FTC

if f is continuous on $[a, b]$ and if F is the derivative of F , i.e. $F' = f$ then

$$\int_a^b f(t) dt = F(b) - F(a)$$

Let $G(x) = \int_a^x f(t) dt$ according to the 2nd FTC, $G'(x) = f(x)$.
but we assumed $F'(x) = f(x)$ so we have two f 's with the same derivative. This means F and G can differ by, at most, a constant C . (pg 174 corollary 2)

So $G(x) = F(x) + C$ i.e. $G(b) = F(b) + C$ and
 $G(a) = F(a) + C$.

So then we know

$$\begin{aligned} G(b) - G(a) &= F(b) + c - [F(a) + c] \\ &= F(b) - F(a) \end{aligned}$$

we know $G(a) = 0$ because $G(x) = \int_a^x f(\tau) d\tau$ which

would become

$$G(a) = \int_a^a f(\tau) d\tau = 0$$

also $G(b) = \int_a^b f(\tau) d\tau$ so

$$\int_a^b f(\tau) d\tau - 0 = F(b) - F(a)$$

$$\int_a^b f(\tau) d\tau = F(b) - F(a) \quad Q.E.D.$$

$$\textcircled{1} \int_3^3 x^3 dx = 0$$

$$\textcircled{2} \int_4^4 \sin x dx = 0$$

$$\textcircled{3} \int_{\frac{\pi}{2}}^0 \sin x dx = -\cos 0 + \cos \frac{\pi}{2} = -1$$

$$\textcircled{4} \int_2^1 x^2 dx = \frac{1^3}{3} - \frac{2^3}{3} = \frac{1}{3} - \frac{8}{3} = -\frac{7}{3}$$

$$\textcircled{5} \text{ (using Lemma 2) } b-a=4 \therefore$$

$$a) \quad 3.4 \leq \int_a^b f(x) dx < 5.4$$

$$12 \leq \int_a^b f(x) dx \leq 20$$

b) The total mass of the string is between 12 and 20.

$$\textcircled{6} \int_1^3 x dx = \bar{X}(3-1)$$

$$\int_1^3 x dx = \frac{3^2}{2} - \frac{1^2}{2} = \frac{8}{2} = 4 \quad \text{so}$$

$$4 = \bar{X}(3-1) \Rightarrow \bar{X} = 2.$$

$$\textcircled{7} \int_0^1 x^2 dx = \bar{X}^2(1-0)$$

$$\int_0^1 x^2 dx = \frac{1^3}{3} - 0 \quad \frac{1}{3} = \bar{X}^2 \therefore \bar{X} = \pm \frac{\sqrt{3}}{3}$$

Note Since $\bar{X}$ is between 0, 1 i.e. $0 < \bar{X} < 1$

$$\bar{X} > 0 \therefore \bar{X} = +\frac{\sqrt{3}}{3}$$

$$\textcircled{8} \int_0^{2\pi} \cos x \, dx = \cos \bar{X} (2\pi - 0)$$

$$\int_0^{2\pi} \cos x \, dx = \sin 2\pi - \sin 0 = 0$$

$$\cos \bar{X} (2\pi) = 0 \Rightarrow \bar{X} \in \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\}$$

$$\textcircled{9} \int_2^4 \frac{1}{x} \, dx = \frac{1}{\bar{X}} (4-2)$$

$$\int_2^4 \frac{1}{x} \, dx = \ln 4 - \ln 2$$

$$\frac{2}{\bar{X}} = \ln\left(\frac{4}{2}\right)$$

$$\frac{2}{\bar{X}} = \ln 2 \Rightarrow \bar{X} = \frac{2}{\ln 2}$$

$$\textcircled{10} e^x \sin 2x \quad \text{let } f = e^x (a \cos 2x + b \sin 2x)$$

$$f' = e^x (a \cos 2x + b \sin 2x) + 2e^x (a \sin 2x + b \cos 2x)$$

$$e^x \sin 2x \stackrel{\text{let}}{=} a e^x \cos 2x + 2b e^x \cos 2x + b e^x \sin 2x + 2a e^x \sin 2x$$

$$= (a+2b) e^x \cos 2x + (b-2a) e^x \sin 2x$$

$$\text{So } a+2b=0 \quad \& \quad b-2a=1$$

$$4b+2a=0$$

$$\frac{5b}{5} = 1$$

$$b = \frac{1}{5} \Rightarrow a + \frac{2}{5} = 0$$

$$a = -\frac{2}{5}$$

$$\therefore f = e^x \left(-\frac{2}{5} \cos 2x + \frac{1}{5} \sin 2x \right)$$

$$(11) f' = \ln x \quad \text{let } f = x \ln x$$

$$f' = 1 + \ln x \quad \text{So,}$$

$$\text{let } f = x \ln x - x \quad \text{then } f' = 1 + \ln x - 1 = \ln x,$$

$$(12) f' = x \sin x^2$$

$$\text{try } f = \sin x^2$$

$$\text{now } f' = 2x \cos x^2$$

$$\text{try } f = -\frac{1}{2} \cos x^2 \quad \text{now}$$

$$f' = x \sin x^2,$$

$$(15) \text{ find } \frac{d^2 y}{dx^2} \quad y = f(x) = \int_0^x (1+t^2) dt,$$

$$y' = (1+x^2) \quad (\text{using 2nd fund thm})$$

$$y'' = 2x$$

?

(16)

$$\lim_{t \rightarrow 1} \frac{\int_1^t e^x dx}{\int_1^t e^{x^2} (1+x) dx}$$

 $\frac{0}{0}$ So use l'Hopital's Rule.

$$\text{Product rule } u = \int_1^t e^{x^2} dx$$

$$du = e^{t^2}$$

$$v = (1+x) \therefore dv = 1$$

$$\lim_{t \rightarrow 1} \frac{e^t}{e^{t^2}(1+t) + \int_1^t e^{x^2} dx}$$

$$\frac{e}{e^1(2)+0} = \frac{e}{2e} = \frac{1}{2}$$

now

(17)

a) area under $|v(t)|$ would mean the distance traveled from t_1 to t_2

b) if the slope is positive it would be the acceleration if negative deceleration.

$$17) \lim_{x_2 \rightarrow x_1} \frac{\int_0^{x_2} e^{t^2} dt - \int_0^{x_1} e^{t^2} dt}{x_2 - x_1}$$

Let $f(x) = \int_0^x e^{t^2} dt$ then $f'(x_1) = e^{x_1^2} = e^{x_1^2}$

$$18) \lim_{h \rightarrow 0} \frac{\int_1^{x+h} \sqrt{\sin t} dt - \int_1^x \sqrt{\sin t} dt}{h}$$

$f(x) = \int_1^x \sqrt{\sin t} dt$ $f'(x) = \sqrt{\sin x}$

2nd FTC:

$$G(x) = \int_a^x f(t) dt \Rightarrow G'(x) = f(x)$$

Consider

$$G(x) = \int_a^{x^2} f(t) dt \quad \text{let } u = x^2$$
$$\frac{du}{dx} = 2x$$

$$G(u) = \int_a^u f(t) dt \quad \text{Also } \frac{d(G(u))}{dx} = \frac{dG(u)}{du} \cdot \frac{du}{dx}$$

$$\text{So } G'(x) = f(x^2) \cdot 2x$$
$$= 2x f(x^2)$$

Example 3 pg 291 implies upper bound "Just drops into $f(t) dt$ " No matter what $f(t)$ actually is.

$$G(x) = \int_a^{x^2} t^3 dt \quad u = x^2 \quad \frac{du}{dx} = 2x$$

$$G'(x) = (x^2)^3 \cdot 2x$$
$$2x^7$$

$$\#16 \quad \lim_{\Delta x \rightarrow 0} \frac{\int_2^{3+\Delta x} x^3 dx - \int_2^3 x^3 dx}{\Delta x}$$

if $f(t) = \int_2^t x^3 dx$ we want def of der
then evaluated at $t=3$

$$\lim_{\Delta x \rightarrow 0} \frac{\int_2^{t+\Delta x} x^3 dx - \int_2^t x^3 dx}{\Delta x} \quad \text{def of der}$$

Then put 3 in for the upper limit t ,
to evaluate and we have original

So using 2nd FTC

$$d(f(t)) = d\left(\int_2^t x^3 dx\right) \Big|_{t=3} =$$

$$t^3 \Big|_{t=3} = 27.$$

For Problems 1-12 2 integrals are requested. Choose any constant values you wish.

pg 300

$$(1) \int \frac{1}{x} dx = \ln|x| + C$$

$$(2) \int e^{4x} dx = \frac{1}{4} e^{4x} + C$$

$$(3) \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

$$(4) \int 3x^2 + 4x^3 dx = x^3 + x^4 + C$$

$$(5) \int 2x^2 dx = \frac{2}{3} x^3 + C$$

$$(6) \int \frac{3 dx}{1+x^2} = 3 \tan^{-1} x + C$$

$$(7) \int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + C$$

$$(8) \int \frac{dx}{x^2} = -\frac{1}{x} + C$$

$$(9) \int \frac{5 dx}{x\sqrt{x^2-1}} = 5 \sec^{-1} x + C \quad |x| > 1$$

$$(10) \int \sec^2 3x dx = \frac{1}{3} \tan 3x + C$$

$$(11) \int \tan 2x \sec^2 2x dx = \frac{1}{2} \sec 2x + C$$

$$(12) \int \frac{x dx}{1+5x^2} = \frac{1}{10} \ln(1+5x^2) + C$$

$$(13) \int x^4 dx = \frac{x^5}{5} + C$$

$$(14) \int \frac{1}{x^4} dx = -\frac{3}{x^3} + C$$

$$(15) \int \sin 2x dx = -\frac{1}{2} \cos 2x + C$$

$$(16) a) (\int x^5 dx)' = x^5 \quad b) (\int \sin x^5 dx)' = \sin x^5$$

(17) $\int_0^{\frac{\pi}{2}} \sin x \, dx = -\int \sin x \, dx \Big|_0^{\frac{\pi}{2}}$ is guaranteed by FTC

(18) a) $\int x^2 \, dx \Big|_1^2 = \frac{x^3}{3} \Big|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$

b) $\int \frac{1}{x} \, dx \Big|_1^2 = \ln x \Big|_1^2 = \ln 2 - \ln 1 = \ln 2$

c) $\int \frac{1}{1+x^2} \, dx \Big|_1^2 = \tan^{-1} x \Big|_1^2 = \tan^{-1} 2 - \tan^{-1} 1 = \tan^{-1} 2 - \frac{\pi}{4}$

(19) a) $\int \cos x \Big|_0^{\frac{\pi}{4}} = \sin x \Big|_0^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2} - 1$

b) $\int \frac{x}{1+x^2} \, dx \Big|_0^1 = \frac{1}{2} \ln(1+x^2) \Big|_0^1 = \frac{1}{2} \ln 2 - \frac{1}{2} \ln 1 = \frac{1}{2} \ln 2$

c) $\int \frac{1}{\sqrt[3]{x}} \, dx \Big|_1^2 = \frac{3}{2} x^{2/3} \Big|_1^2 = \frac{3}{2} (\sqrt[3]{4} - 1)$

(20) a) $x^2 \Big|_1^2 = 4 - 1 = 3$

b) $x^2 \Big|_{-1}^1 = 1 - 1 = 0$

c) $\sin x \Big|_0^{\pi} = 0 - 0 = 0$

d) $\cos x \Big|_0^{\pi} = -1 - 1 = -2$

(21) $\int 1 \, dx = x + C$

(22) a) $\int \frac{1+9x^2}{x} \, dx = \int \frac{1}{x} \, dx + \int 9x \, dx = \ln|x| + \frac{9}{2}x^2 + C$

b) $\int \frac{x}{1+9x^2} \, dx = \frac{1}{18} \ln(1+9x^2) + C$

- (23) a) $\int x^2 dx$ is a fctn b) $\int x^2 dx|_1^3$ is a number
 c) $\int_1^3 x^2 dx$ is a number

24 a) $\int x^2 dx|_1^3$ is not a limit of sums, this is the evaluation of a fctn at 2 values,
 $\int_1^3 x^2 dx$ is the limit of sums.

b) The two numbers are equal as per the FTC,

(25) e^{-x^2} does have an antiderivative, whether I can find it or not does not control the fact that it exists,

- (26) a) True, an elementary fctn has an elementary derivative
 b) False, an elementary fctn may or may not have an elementary antiderivative,

(27) $\int x^2 \sin ax dx = \frac{2x}{a^2} \sin ax + \frac{2}{a^3} \cos x - \frac{x^2}{a} \cos ax.$

Order of right side,

(28) $\int x \sin ax dx = \frac{x^2}{4} - \frac{x \sin 2ax}{4a} - \frac{\cos ax}{8a^3}$
 $\frac{1}{2}x - \left[\frac{1}{4a} \sin 2ax - \frac{x \cos 2ax}{2} + \frac{\sin ax}{8a^2} \right]$
 $\frac{1}{2} \sin ax \cos ax$

(30)

$$\frac{1}{2} \sin^4 \theta$$

$$2 \sin^3 \theta \cos \theta$$

$$\sin^2 \theta \cos^2 \theta$$

$$\sin^2 \theta$$



$$(29) \quad f(\theta) = \sin^2 \theta \sin 2\theta$$

$$f'(\theta) = 2\sin \theta \cos \theta \sin 2\theta + 2\sin^2 \theta \cos 2\theta$$

$$= 2\sin 2\theta + 2\sin^2 \theta \cos 2\theta \stackrel{?}{=} 0$$

$$\sin 2\theta + \sin^2 \theta \cos 2\theta = 0$$

$$2\sin \theta \cos \theta + \sin^2 \theta \cos 2\theta = 0$$

$$\sin \theta (2\cos \theta + \sin \theta \cos 2\theta) =$$

$$(2\cos \theta + \sin \theta (2\cos^2 \theta - 1)) = 0$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$2\sin \theta (\cos \theta \sin 2\theta + \sin \theta \cos 2\theta) \stackrel{?}{=} 0$$

$$2\sin \theta (\sin(\theta + 2\theta)) = 0$$

$$2\sin \theta (\sin 3\theta) = 0$$

$$2\sin \theta \sin 3\theta = 0$$

$$\sin \theta \sin 3\theta = 0 \quad \{K\pi\}$$

$$\theta = 0, \pi$$

$$3\theta = 0, \pi$$

$$2K\pi \pm \frac{\pi}{3}$$

$$\theta = 0, \frac{\pi}{3}$$

$$\therefore \theta \in \left\{ \frac{K\pi}{3} + 2K\pi, K\pi \right\}$$

$$\left\{ \frac{(6K+1)\pi}{3}, K\pi \right\}$$

$$(30) \quad f' = \sin^2 \theta \sin 2\theta$$

$$\sin(\sin^2 \theta) = 2\sin \theta \cos \theta$$

OR

$$\sin 2\theta = 2\sin \theta \cos \theta$$

$$\text{so } f' = \sin^2 \theta \cos \theta$$

$$\text{Then } (\sin^2 \theta)^2 \text{ would be}$$

$$(\sin^2 \theta)' = 2\sin \theta \cos \theta$$

$$2a \sin \theta \cos \theta + 2b \cos \theta$$

$$2(\sin^2 \theta) 2\sin \theta \cos \theta$$

$$2\sin^2 \theta \sin 2\theta$$

$$\text{Let } f = \frac{1}{2} (\sin^2 \theta)^2$$

has the

$$\text{So } f = \frac{1}{2} \sin^2 \theta \quad \frac{d(\sin^2 \theta)}{d\theta}$$

some adjustments.

$$\sin^2 \theta (2\sin \theta \cos \theta) =$$

$$\sin^2 \theta \sin 2\theta$$

too hard.

$$(3) \left[\int_0^1 f(x) dx \right]^2 \geq \int_0^1 [f(x)]^3 dx$$

$$f(0) = 0 \\ 0 < f'(x) \leq 1$$

using hint let $0 \leq \tau \leq 1$

$$\text{let } g(\tau) = \left[\int_0^\tau f(x) dx \right]^2 - \int_0^\tau [f(x)]^3 dx \quad \text{so we can show } g(\tau) \geq 0 \text{ for } 0 \leq \tau \leq 1$$

$$g'(\tau) = 2 \left(\int_0^\tau f(x) dx \right) \overset{u^2}{\downarrow} f(\tau) - \overset{du}{\downarrow} (f(\tau))^3 \\ = 2f(\tau) \left[2 \int_0^\tau f(x) dx - (f(\tau))^2 \right]$$

This is ≥ 0 $\therefore$ we must check other factor.

$$\text{let } h = 2 \int_0^\tau f(x) dx - (f(\tau))^2$$

$$\text{Then } h' = f(\tau) - 2f(\tau)f'(\tau) \\ 2f(\tau)(1 - f'(\tau))$$

$$\uparrow \quad \text{Since } f'(\tau) < 1, 1 - f'(\tau) > 0$$

So h is non decreasing $\therefore g' = f(\tau) \cdot h(\tau)$ is not negative $\therefore g$ is also non decreasing.

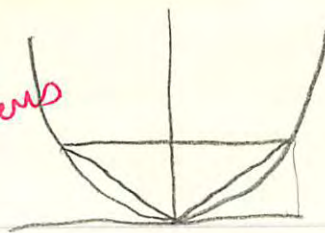
So $g(\tau) \geq g(0)$ which $= 0$ (because $f(0) = 0$).

Now if we let $\tau = 1$ we have the original particular solution.

$$b) \left(\int_0^1 dx \right)^2 \stackrel{?}{=} \int_0^1 (1)^3 dx \quad \left(\int_0^1 x dx \right)^2 \stackrel{?}{=} \int_0^1 x^3 dx \\ \left(x \Big|_0^1 \right)^2 = 1 \times 1 = 1 \quad \left(\frac{x^2}{2} \Big|_0^1 \right)^2 = \frac{x^4}{4} \Big|_0^1 \\ \frac{1}{4} = \frac{1}{4}$$

Discuss

(32)



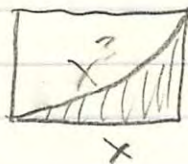
What does "between" mean?

*

Area of Triangle
area of Sector.

base = $2x$ height $x^2 \therefore x^3 = \text{Area}$.

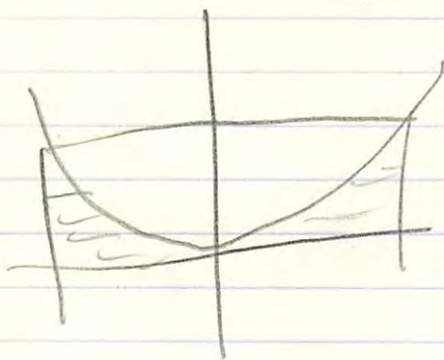
Book answer
 $\frac{3}{4}$



x^2 required area is $x^3 - \text{Shaded area}$
Shaded area $\int_0^x x^2 dx = \frac{x^3}{3}$

So area requested = $x^3 - \frac{x^3}{3} = \frac{2x^3}{3}$ half of req area,
 $\frac{4x^3}{3}$ shaded area.

$$\lim_{x \rightarrow 0} \frac{x^3}{\frac{4x^3}{3}} = \frac{3}{4}$$



Area $2x^3$

$$2x^3 - 2 \int_0^x x^2 dx$$

$$2x^3 - \frac{2x^3}{3} = \frac{4x^3}{3}$$

lim

Area of $\Delta \frac{1}{2} 2x \cdot x^2 = x^3$

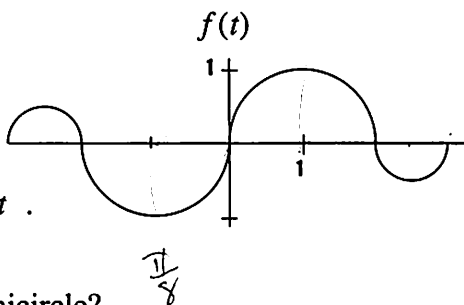
$$\text{So } \lim_{x \rightarrow 0} \frac{x^3}{\frac{4x^3}{3}} = \frac{3}{4} \quad \begin{matrix} \text{area of } \Delta \\ \text{area of } \\ \text{Parabola} \end{matrix}$$

$$\int_0^x t^2 dt = \left. \frac{t^3}{3} \right|_0^x = \frac{x^3}{3}$$

But angle $x \cdot x^2 = x^3 \therefore$
So $2 \cdot \left(x^3 - \frac{x^3}{3} \right) = \frac{4x^3}{3}$ area
in Parabola below line.

The graph of the function f on the right consists of four semicircles.

Given: $g(x) = \int_0^x f(t) dt$, $h(x) = \int_1^x f(t) dt$ and $j(x) = \int_0^{2x} f(t) dt$.



- I
 - a) Consider a circle of radius one-half. What is the area of its semicircle?
 - b) Consider a circle of radius one. What is the area of its semicircle? $\frac{\pi}{2}$
 - c) Is the function $f(t)$ odd, even, or neither?

- II
 - a) How does the answer to part (Ic) effect $g(x)$?
 - b) Find: $g(0)$, $g(2)$, $g(3)$, $g(-2)$, $g(-3)$
 - c) Make a number line for $g'(x)$. Mark zeros as well as indicate positive and negative intervals.
 - d) Specifically state the intervals when $g(x)$ is increasing; decreasing.
 - e) Determine the *coordinates* of all extrema of $g(x)$. Justify, of course.
 - f) Determine the *coordinates* of all Points of Inflection of $g(x)$.
 - g) Sketch $g(x)$.
 - h) For what values of x is $g(x)$ non-negative.

- III
 - a) What is the domain of $h(x)$?
 - b) Compute: $h(1)$, $h(0)$, $h(2)$, $h(3)$, $h(-2)$, $h(-3)$, $h(-1)$
 - c) Draw a $h'(x)$ number line. Indicate $+$, $-$ as appropriate.
 - d) Compute *coordinates* of extrema. Justify, of course.
 - e) Draw a $h''(x)$ number line. Indicate $+$, $-$ as appropriate.
 - f) Compute *coordinates* of PI.
 - g) Sketch $h(x)$.

- IV
 - a) What is the domain of $j(x)$?
 - b) As in section III, compute the significant values of $j(x)$.
 - c) Draw $j'(x)$ number line, Indicate $+$, $-$ as appropriate.
 - d) Compute *coordinates* of extrema. Justify, of course.
 - e) Draw $j''(x)$ number line, Indicate $+$, $-$ as appropriate.
 - f) Compute *coordinates* of PI.
 - g) Sketch $j(x)$.

I a) $\frac{\pi}{8}$

I b) $\frac{\pi}{2}$

I c) f is an odd function because the function reflects through the origin.
For example, $(1,1)$ reflects to $(-1,-1)$.

II a) g will be an even function so $g(a) = g(-a)$.

II b) $g(0) = 0$ because $g(0) = \int_0^0 f(t) dt$.

$$g(2) = \int_0^2 f(t) dt \text{ which is } \frac{\pi}{2} \text{ (from part b) so } g(-2) = g(2) = \int_0^2 f(t) dt = \frac{\pi}{2}$$

$$g(3) = \int_0^3 f(t) dt = \frac{\pi}{2} - \frac{\pi}{8} = \frac{3\pi}{8} \text{ subtracting the smaller semicircle from the larger so}$$

$$g(-3) = g(3) = \int_0^3 f(t) dt = \frac{\pi}{2} - \frac{\pi}{8} = \frac{3\pi}{8}.$$

II c) $\frac{+}{-3} \frac{0}{-2} \frac{-}{0} \frac{+}{2} \frac{0}{3} \frac{-}{3}$ *G' is $\neq 0$ when f above x axis.*

II d) $g'(x) > 0$ for $-3 < x < -2$ and for $0 < x < 2$ thus g is increasing in these intervals.
 $g'(x) < 0$ for $-2 < x < 0$ and for $2 < x < 3$ thus g is decreasing in these intervals.

II e) $g(-3) = g(3) = \frac{3\pi}{8}$. The left end point is a minimum because g increases from it and the right end point

is a minimum because g decreases to it. coordinates: $\left(3, \frac{3\pi}{8}\right)$ and $\left(-3, \frac{3\pi}{8}\right)$

$g(-2) = g(2) = \frac{\pi}{2}$ are maximums by $f'(x)$ test. coordinates: $\left(2, \frac{\pi}{2}\right)$ and $\left(-2, \frac{\pi}{2}\right)$

$g(0) = 0$ by $f'(x)$. coordinates $(0, 0)$

II f) PI occur in g when $g''(x)$ changes sign. This is when $f'(x)$ changes sign. From the our graph, this is when f changes from increasing to decreasing (or vice versa).

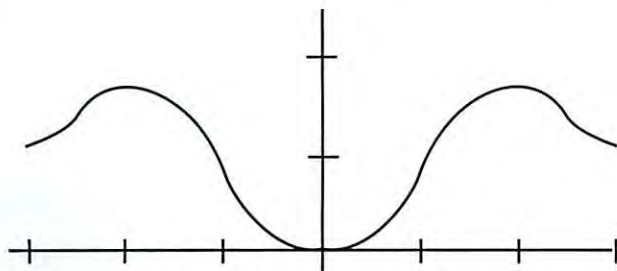
PI occur when x is an element of the set: $\{-2.5, -1, 1, 2.5\}$. Note: $\frac{\pi}{2} \cdot \frac{1}{2} = \frac{\pi}{4}$ and $\frac{7\pi}{16} = \frac{\pi}{2} - \frac{\pi}{16}$

Coordinates: $\left(-2.5, \frac{7\pi}{16}\right), \left(-1, \frac{\pi}{4}\right), \left(1, \frac{\pi}{4}\right), \left(2.5, \frac{7\pi}{16}\right)$

$\frac{up}{-3} \frac{dn}{-2.5} \frac{-}{-2} \frac{0}{-1} \frac{up}{0} \frac{dn}{1} \frac{up}{2} \frac{dn}{2.5} \frac{up}{3}$

Note: $g''(x)$ DNE for $x \in \{-2, 0, 2\}$ because the tangent line to the graph of f becomes vertical.

II g)



II h) g is non-negative for all $x \in [-3, 3]$.

III a) $D_h x \in [-3, 3]$ There is no change because upper bound is x (rather than $x+1$, for example)

III b) $h(1) = \int_1^1 f(t) dt = 0 \quad (1, 0) \quad h(0) = \int_1^0 f(t) dt = -\int_0^1 f(t) dt = -\frac{\pi}{4} \quad \left(0, -\frac{\pi}{4}\right)$

$h(2) = \int_1^2 f(t) dt = \frac{\pi}{4} \quad \left(2, \frac{\pi}{4}\right) \quad h(3) = \int_1^3 f(t) dt = \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8} \quad \left(3, \frac{\pi}{8}\right)$

$h(-1) = \frac{\pi}{4} - \frac{\pi}{4} = 0 \quad (-1, 0) \quad h(-2) = -\left(-\frac{\pi}{4}\right) = \frac{\pi}{4} \quad \left(-2, \frac{\pi}{4}\right)$

$h(-3) = \text{this is } h(-2) - \frac{\pi}{4} = \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8} \quad \left(-3, \frac{\pi}{8}\right)$

III c) This is the same as it was for II c because $h'(x) = f(x)$ just as $g'(x) = f(x)$.

$$\frac{0+0-0-0+0-0}{-3-2-0-2-3}$$

III d) $\left(-3, \frac{\pi}{8}\right)$ min. end pt increasing from it. $\left(-2, \frac{\pi}{4}\right)$ max $f'(x)$ test

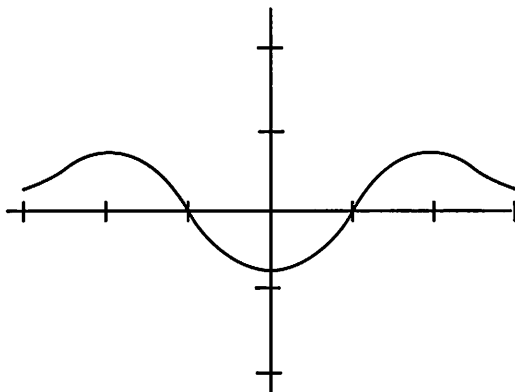
$\left(0, -\frac{\pi}{4}\right)$ min $f'(x)$ test $\left(2, \frac{\pi}{4}\right)$ max $f'(x)$ test $\left(3, \frac{\pi}{8}\right)$ min end pt decrease to it.

III e) $h''(x)$ $\frac{\text{up} \quad \text{dn} \quad \text{up} \quad \text{dn} \quad \text{up}}{\frac{+0-0-0+0-0}{-3-2.5-2-1-0-1-2-2.5-3}}$ Note: This is exactly the same as $g''(x)$ (II f)

III f) As in part II f, the x values of the PI are the same.

$-\frac{\pi}{4} + \frac{\pi}{2} - \frac{\pi}{16} = \frac{3\pi}{16} \quad \left(-\frac{5}{2}, \frac{3\pi}{16}\right) \quad \frac{\pi}{4} - \frac{\pi}{16} = \frac{3\pi}{16} \quad \left(\frac{5}{2}, \frac{3\pi}{16}\right) \quad (-1, 0), (1, 0)$

III g) $h(x)$:



✓ a) $-3 \leq t \leq 3$ so
 $-3 \leq 2x \leq 3$ $D_j \ x \in \left[-\frac{3}{2}, \frac{3}{2}\right]$
 $-\frac{3}{2} \leq x \leq \frac{3}{2}$

IV b) The $2x$ just changes the period of the function much in the same manner as $\sin x$ compared to $\sin 2x$.

$$j(0) = 0, \quad j(-1) = j(1) = \int_0^{2 \cdot 1 = 2} f(t) dt = \frac{\pi}{2}, \quad j\left(-\frac{3}{2}\right) = j\left(\frac{3}{2}\right) = \int_0^{2 \cdot \frac{3}{2} = 3} f(t) dt = \frac{3\pi}{8}$$

$$j\left(-\frac{1}{2}\right) = j\left(\frac{1}{2}\right) = \int_0^{2 \cdot \frac{1}{2} = 1} f(t) dt = \frac{\pi}{4}, \quad j\left(-\frac{5}{4}\right) = j\left(\frac{5}{4}\right) = \int_0^{2 \cdot \frac{5}{4} = \frac{5}{2}} f(t) dt = \frac{\pi}{2} - \frac{\pi}{16} = \frac{7\pi}{16}$$

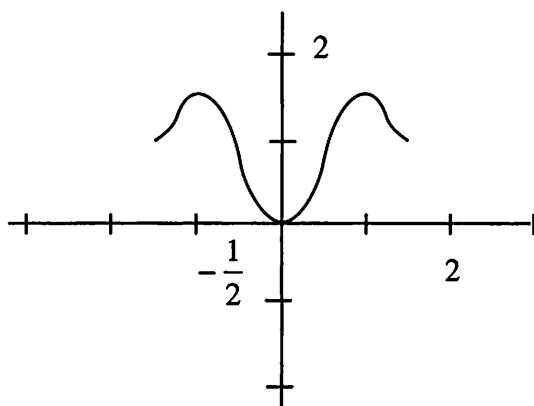
IV c) $j'(x)$ $\frac{[+ \ 0 \ - \ 0 \ + \ 0 \ -]}{-\frac{3}{2} \quad -1 \quad \hat{0} \quad \frac{1}{4} \quad 1 \quad \frac{3}{2}}$ $j'\left(\frac{1}{4}\right)$ would be the slope of $f(t)$ at $x = \frac{1}{2}$.
 From the original picture of f , f is clearly increasing. The $+$, $-$ signs alternate from there.

IV d) $\left(-\frac{3}{2}, \frac{3\pi}{8}\right)$ is min. end pt, increasing from it $\left(-1, \frac{\pi}{2}\right)$ is max. f' test. $(0, 0)$ min f' test.
 $\left(1, \frac{\pi}{2}\right)$ max, f' test. $\left(\frac{3}{2}, \frac{3\pi}{8}\right)$ is min. end pt. decreasing to it.

IV e) $\frac{\text{up} \quad \text{dn} \quad \text{up} \quad \text{dn} \quad \text{up}}{[+ \ 0 \ - \ 0 \ + \ 0 \ -]} \frac{[+ \ 0 \ - \ 0 \ + \ 0 \ -]}{-\frac{3}{2} \quad -\frac{5}{4} \quad -1 \quad -\frac{1}{2} \quad \hat{0} \quad \frac{1}{2} \quad 1 \quad \frac{5}{4} \quad \frac{3}{2}}$

IV f) The coordinates are listed in IV b.

IV g)



8. Let $f(x) = \sin x$ and $g(x) = x^2$. Graph f and g . Which of the following points best approximates the intersection of the two graphs for $x > 0$? a) (2, 1) b) (.9, .9) c) (4, -1)

9. Find all solutions of $\sqrt{x^3} + x \leq -1$ by using a graph. *There are no solutions x cannot be neg.*

10. Describe or sketch the graph of $f(x) = \ln(x + \sqrt{x^2 - 4}) + \ln(x - \sqrt{x^2 - 4})$. *Domain $x \geq 2$ Hyperbolic line*

11. Graph $f(x) = 2x^{1/2} - x^2$. Which of the following appears to be nearest any zeros of f ? a) -2 b) 2 c) .5 d) 1.58

12. Let $f(x) = \frac{1}{1 + \cos x^{-2}}$. Approximate $f(1.1)$ to 4 decimal places. *.5961*

13. Let $f(x) = 3x^{1/2} + x^{2/3}$. Approximate $f(1.3)$ to 4 decimal places. *4.6117*

14. Find all solutions of $x^4 + 3x + 5 > 0$. Express your answer as an interval or union of intervals. *none*

15. Graph $f(x) = 1 + \sqrt{x}$, $g(x) = 1 + \sqrt{1-x}$ and $h(x) = (f+g)(x)$. ie $f(x) + g(x)$. Describe the graph of h . *$J(x) = f(g(x)) = f \circ g$ Then $J(x) = 1$*

16. Graph $f(x) = x^5 + 4x^2 + x$ using a range of $[-4, 4]$ on both axes. Which, if any, of the following appear to be nearest the zeros of f ?

- a) -2 b) -1.5 c) 0 d) 1

17. Describe and sketch the graph of $f(x) = \ln(3x) - \ln x$. *90 degrees rotated. $-\ln 3$ horizontal line*

18. Two particles start at the origin and move along the x -axis. For $0 \leq t \leq 10$, their respective position functions are given by $x_1 = \sin t$ and $x_2 = e^{-2t} - 1$. For how many values of t do the particles have the same velocity? *Cost $\frac{d}{dt} = -2e^{-2t}$ as $t \rightarrow \infty$ becomes inf*

- a) none b) one c) two d) three e) four

19. If c satisfies the conclusion of the Mean Value Theorem for $f(x) = x^4 - 3x^3 + 2x - 3$ on the interval $-1 \leq x \leq 2$, then c could be

- a) -0.59 only b) 0.00 only c) 0.84 only d) -0.59 and 0.84 e) -0.43 and 0.54

20. A car with constant acceleration goes from 0 to 88 feet per second in 440 feet. How long does it take?

Solve with Solver
 $\frac{dv}{dt} = k$
 $\int dv = \int k dt$
 $v = kt + c$
 $v(0) = 0 \therefore c = 0$
 $v(t) = kt = 88$
 $s = \int v dt$
 $s = \frac{k}{2}t^2 + D$
 $s(0) = 0 \therefore D = 0$
 $\frac{1}{2}kt^2 = 440$
 $\frac{1}{2} \cdot 88 \cdot t = 440$
 $t = 10$

8. Let $f(x) = \sin x$ and $g(x) = x^2$. Graph f and g . Which of the following points best approximates the intersection of the two graphs for $x > 0$? a) $(2, 1)$ b) $(.9, .9)$ c) $(4, -1)$

9. Find all solutions of $\sqrt{x^3} + x \leq -1$ by using a graph.

10. Describe or sketch the graph of $f(x) = \ln(x + \sqrt{x^2 - 4}) + \ln(x - \sqrt{x^2 - 4})$.

11. Graph $f(x) = 2x^{1/2} - x^2$. Which of the following appears to be nearest any zeros of f ?

- a) -2 b) 2 c) $.5$ d) 1.5

12. Let $f(x) = \frac{1}{1 + \cos x^{-2}}$. Approximate $f(1.1)$ to 4 decimal places.

13. Let $f(x) = 3x^{1/2} + x^{2/3}$. Approximate $f(1.3)$ to 4 decimal places.

14. Find all solutions of $x^4 + 3x + 5 > 0$. Express your answer as an interval or union of intervals.

15. Graph $f(x) = 1 + \sqrt{x}$, $g(x) = 1 + \sqrt{1 - x}$ and $h(x) = (f + g)(x)$. ie $f(x) + g(x)$ and $k(x) = f(g(x))$. ie $f \circ g$. Describe the graphs of h and k .

16. Graph $f(x) = x^5 + 4x^2 + x$ using a range of $[-4, 4]$ on both axes. Which, if any, of the following appear to be nearest the zeros of f ?

- a) -2 b) -1.5 c) 0 d) 1

17. Describe and sketch the graph of $f(x) = \ln(3x) - \ln x$.

18. Two particles start at the origin and move along the x -axis. For $0 \leq t \leq 10$, their respective position functions are given by $x_1 = \sin t$ and $x_2 = e^{-2t} - 1$. For how many values of t do the particles have the same velocity?

- a) none b) one c) two d) three e) four

19. If c satisfies the conclusion of the Mean Value Theorem for $f(x) = x^4 - 3x^3 + 2x - 3$ on the interval $-1 \leq x \leq 2$, then c could be

- a) -0.59 only b) 0.00 only c) 0.84 only
d) -0.59 and 0.84 e) -0.43 and 0.54

A car with constant acceleration goes from 0 to 88 feet per second in 440 feet. How long does it take?

Calculator 101

1. Let f be the function given by $f(x) = \cos x$. If the number c satisfies the conclusion of the Mean Value Theorem (also known as the Law of the Mean) for f on the closed interval $[2, 4]$, then $c =$
a) 0.119 b) 2.902 c) 3.023 d) 3.142 e) 3.261
2. The rate of decay of a radio active substance is proportional to the amount of the substance present at any time t . In 1840 there were 50 grams of the substance and in 1910 there were 35 grams. To the nearest gram, how many grams of the substance remained in 1990?
a) 18 b) 20 c) 23 d) 36 e) 107
3. If $f(x) = \sqrt[5]{x^3 - 2x}$, then $f'(\sqrt{3}) =$
a) 0.129 b) 0.139 c) 0.902 d) 0.906 e) 1.116
4. The slope of the line containing the point $(4, 0)$ and tangent to the graph of $y = e^{-x}$ is
a) 0.050 b) 0.018 c) -0.050 d) -0.018 e) -0.070
5. If $f(x) = \ln x$, then the average rate of change of f on the interval $[2, 5]$ is
a) 0.100 b) 0.133 c) 0.305 d) 1.155 e) 1.220
6. A 16-foot ladder is leaning against a vertical wall. The top end of the ladder is sliding down the wall at the rate of 5.6 feet per second. What is the rate of change, in radians per second, of the angle θ (the angle opposite the vertical wall) at the instant when the top of the ladder is 7 feet above the ground?
a) -6.2 b) -0.8 c) -0.4 d) -0.3 e) -0.1
7. The slope of the line tangent to the curve $y = x^{-x}$ at $x = 0.368$ is closest to the integer
a) -6 b) -1 c) 0 d) 1 e) 6

①

$$f(x) = \cos x$$

$$\frac{f(4) - f(2)}{4 - 2} \stackrel{?}{=} f'(x)$$

$$-0.1187 = -\sin x$$

$$.1187 = \sin x$$

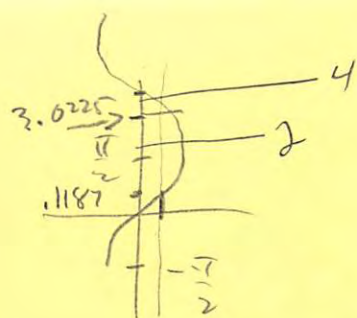
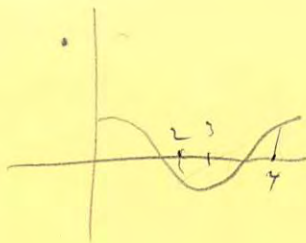
$$\sin^{-1}(.1187) = \sin^{-1}(\sin(\pi - x))$$

$$.119029 = \pi - x$$

$$x = \pi - .119029$$

$$x = 3.0225$$

(C)



$$\sin(\pi - x) = \sin \pi \cos x - \sin x \cos \pi$$

$$= \sin x$$

But .1187 is not in the interval $[2, 4]$

however $\pi - .1187$ is between

2 and 4 AND.

$$\sin(\pi - x) = \sin x$$

②

$$y = A e^{kt}$$

$$T=0 \quad y_1 = 1840$$

$$y = 50 e^{kt}$$

$$\frac{7}{10} \frac{35}{50} = e^{70k}$$

$$\ln .7 = 70k$$

$$\frac{1}{70} \ln .7 = k$$

$$k = -.00509$$

$$y = 50 e^{-.00509t}$$

$$T = 150$$

$$y = 23.3 \text{ "C"}$$

- OR -

$$\left(\frac{7}{10}\right)^{\frac{1}{70}} = e^k = .9949 \dots$$

$$y = 50 (.9949)^T$$

$$= 50 (.465)$$

$$= 23.2829 \dots$$

(3)

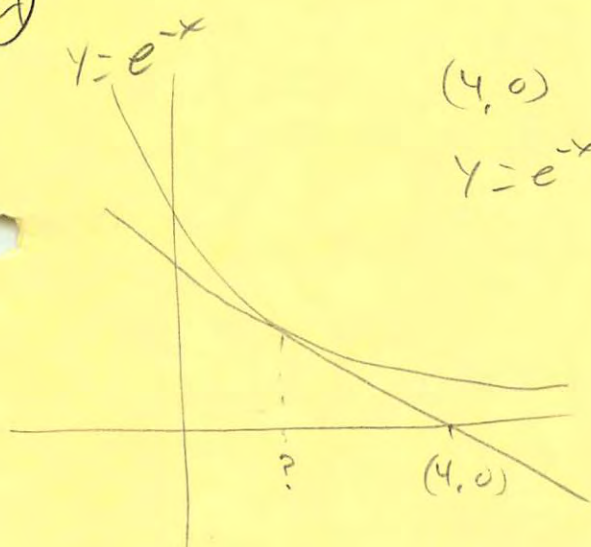
$$f(x) = \sqrt[5]{x^3 - 2x} \quad f'(\sqrt{3}) = ?$$

$$f'(x) = \frac{1}{5} (x^3 - 2x)^{-\frac{4}{5}} (3x^2 - 2)$$

$$= \frac{3x^2 - 2}{5(x^3 - 2x)^{\frac{4}{5}}} = \frac{9 - 2}{5 \cdot 1.3^{\frac{2}{5}}} = \frac{7}{5(1.55)} = .902$$

$$NDeriv(Y_1, X, \sqrt{x}) = .902153 \dots \quad (C)$$

(4)



$(4, 0)$

$$y = e^{-x} \quad y' = -e^{-x}$$

Equation of line

line $\stackrel{\text{Set}}{=}$ curve
to see where
intersection
occurs.

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -e^{-x}(x - 4)$$

$$y = -e^{-x}(x - 4)$$

$$-e^{-x}(x - 4) \stackrel{\text{Set}}{=} e^{-x}$$

$$x - 4 = -1$$

$$x = 3$$

intersection at
 $(3, e^{-3})$

Slope at intersection $-e^{-3}$

$$-0.04978 \approx -0.050 \quad (C)$$

$$\frac{1}{x}$$

$$\frac{\frac{1}{x} - \frac{1}{2}}{\quad}$$

$$\frac{f(b) - f(a)}{b - a} = f$$

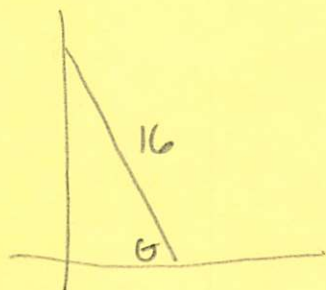
(5) $f(x) = \ln x$ $[2, 5]$

note that this is

$\frac{1}{5-2} \int_2^5 \frac{1}{x} dx$ is average value of $\frac{1}{x}$

$\frac{\ln 5 - \ln 2}{5-2} = .305$ (C)

(6)



$\frac{dy}{dt} = -5.6$

$\frac{d\theta}{dt} = ?$

$\sin \theta = \frac{y}{16}$

$\cos \theta \frac{d\theta}{dt} = \frac{1}{16} \frac{dy}{dt}$

$\frac{d\theta}{dt} = \frac{1}{16} (-5.6) \cdot \frac{1}{.8992}$

$\frac{d\theta}{dt} = -.3892$ (C)

$\sin \theta = \frac{7}{16}$

$\theta = .4528$

$\cos \theta = .8992$

Note calculator must be in Radians.

(7)

$y = x^{-x}$ $x = .368$

$\ln y = -x \ln x$

$\frac{1}{y} \frac{dy}{dx} = -\ln x - 1$

$\frac{dy}{dx} = -x^{-x} (\ln x + 1)$

$x = .368$

$= -.00047$

nearest integer is zero

(C)

① ③ Students

② "ALSO draw the relation's arcsine."

② ③ Students

1. Solve for K
then solve problem

others do not solve for K .
Solve the problem.

③ ② Show both calc work and do problem using the
only for arithmetic work.

④ ④ Draw graph of y . Locate given point. Draw
tangent line. Determine equation of tangent line.
solve problem

⑤ ③ Set up and compute using 1st FTC
compare answers. Explain discrepancies

⑥ ④ after completely Problem, redo with calculator set
to degrees,

⑦ ② Compute the derivative and evaluate. Then
write a single entry on the calculation
which does the entire problem.



2+3

$$y_1 = \sin \tau$$

$$y_2 = e^{-2\tau} - 1$$

$$\text{Ans } \tau = -2e^{-2\tau}$$

Tell what those values of τ are.

Antiderivatives

Students should read Pgs 298-300.
No other comments should be required.

GQ chap 8

① a) every fctn that is continuous over $[a, b]$ has a definite integral over $[a, b]$.

True. There is area w/in bounds.

b) Every fctn that is continuous $\forall x$ is the derivative of some fctn.

True Corollary p. 291

c) Every fctn that is differentiable $\forall x$ is the derivative of some fctn.

True if its differentiable then it is continuous and then we apply "b" above.

③ a) State Both FTC

b) $F(b) - F(a)$ would be distance traveled during the time period $t=a$ to $t=b$.

$$\int_a^b f(t) dt \quad \text{same interpretation as above.}$$

④ a) $\int (x + \frac{3}{\sqrt{x}}) dx = \frac{x^2}{2} + \frac{3}{2} \sqrt{x} + C$

b) $\int e^{x^3} dx$ not elementary.

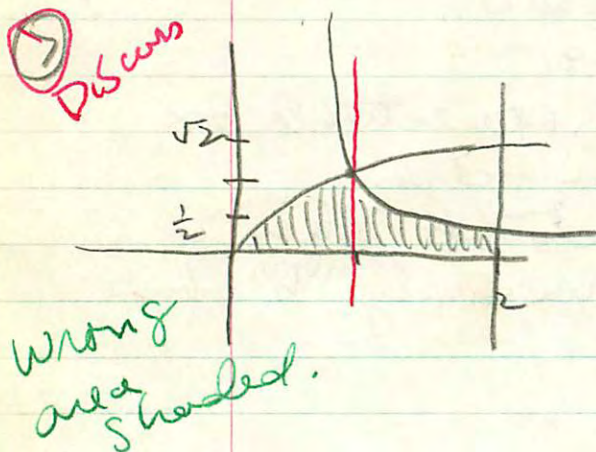
e) $\int e^{-3x} dx = -\frac{1}{3} e^{-3x} + C$

$$(6) a) \int \frac{1}{2x+3} dx = \frac{1}{2} \ln(2x+3) + C$$

$$b) \int \frac{1}{(2x+3)^2} dx = \frac{-1}{2(2x+3)} + C$$

$$c) \int \frac{1}{\sqrt{2x+3}} = \sqrt{2x+3} + C$$

⑦ Discuss

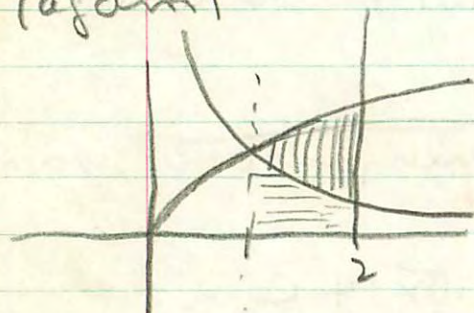


$$\int_0^1 \sqrt{x} dx + \int_1^2 \frac{1}{x} dx$$

$$\frac{2}{3} x^{\frac{3}{2}} \Big|_0^1 + \ln x \Big|_1^2$$

$$\frac{2}{3} + \ln 2,$$

⑦ (again)



$$\int_1^2 \sqrt{x} dx - \int_1^2 \frac{1}{x} dx$$

$$\frac{2}{3} x^{\frac{3}{2}} \Big|_1^2 - \ln 2$$

$$\frac{2}{3} \cdot \sqrt{8} - \frac{2}{3} - \ln 2$$

$$\frac{4\sqrt{2}}{3} - \frac{2}{3} - \ln 2$$

To arrive at text answer of $\frac{4\sqrt{2} - 2 - \ln 8}{3}$
 change $\ln 2$ to be $\ln 8^{\frac{1}{3}} = \frac{1}{3} \ln 8$

$$8a \quad \frac{d}{dx} \left(\int_1^x 3t^2 \sin 2t^3 dt \right) = 3x^2 \sin 2x^3 + C$$

Discuss

(b)

$$\frac{d}{dx} \left(\int_1^{x^3} \sin 2t dt \right) =$$

Notice
use of
chain
Rule.

$$\text{let } u = x^3$$

$$\frac{du}{dx} = 3x^2$$

$$\frac{d}{dx} \left(\int_1^u \sin 2t dt \right) =$$

$$(\sin 2u) \frac{du}{dx} = 3x^2 \sin 2x^3$$

c) obvious. could differ by constant.

(9)

$$a) \quad x^3 \Big|_{-1}^1 = 1 + 1 = 2$$

$$b) \quad x^4 \Big|_{-1}^1 = 1 - 1 = 0$$

$$c) \quad \cos 2x \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = 0 + 0 = 0$$

$$d) \quad e^{-x} \Big|_0^1 = \frac{1}{e} - 1 = \frac{1-e}{e}$$

Try doing it like this:

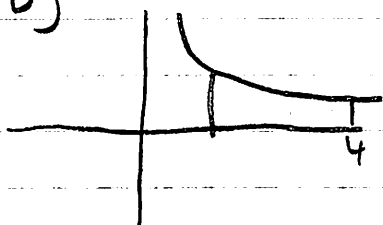
- ① integrate
- ② evaluate at endpoints
- ③ take deriv.

$$1a) y = x^{-1} \quad y' = -x^{-2}$$

$$b) \int \frac{1}{x^2} dx = -x^{-1} + C$$

$$c) \int_1^4 \frac{1}{x^2} dx = -x^{-1} \Big|_1^4 = -\frac{1}{4} + 1 = \frac{3}{4}$$

D)



$$\text{radius of circle} = \frac{1}{x}$$

$$\text{Volume } \pi r^2$$

$$\text{fctn} = \frac{\pi}{x^2}$$

$$\text{Volume } \pi \int \frac{dx}{x^2} = \pi \left(\frac{3}{4} \right) = \frac{3\pi}{4}$$

$$2a) y = \sqrt{1+x^2} \quad y' = \frac{2x}{2\sqrt{1+x^2}} = \frac{x}{\sqrt{1+x^2}}$$

$$b) \int \frac{x dx}{\sqrt{1+x^2}} = \sqrt{1+x^2} + C$$

$$c) \int_0^{\sqrt{3}} \frac{x dx}{\sqrt{1+x^2}} = \sqrt{1+x^2} \Big|_0^{\sqrt{3}} = 2 - 1 = 1$$

D) This was done in c above.

$$E) \int_{\sqrt{3}}^{\sqrt{8}} \frac{t dt}{\sqrt{1+t^2}} = \sqrt{1+t^2} \Big|_{\sqrt{3}}^{\sqrt{8}} = 3 - 2 = 1$$

$$③ a) y = x^5 \quad y' = 5x^4 \quad b) y = \sqrt{x} \quad y' = \frac{1}{2\sqrt{x}}$$

$$c) y = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) \quad d) y = \tan^{-1} x \quad y' = \frac{1}{1+x^2}$$

$$\frac{1}{x+1} + \frac{1}{1-x}$$

$$1-x+x+1$$

$$\frac{2}{1-x^2}$$

$$\textcircled{1} \text{ a) } y = \frac{e^{-x}}{1+2x^2} \quad y' = \frac{-(1+2x^2)e^{-x} - 4xe^{-x}}{(1+2x^2)^2}$$

b) magnification at a value is simply the derivative of the lens evaluated at that value,
 so $y' \Big|_{\bar{x}=1} = \frac{-(1+2)e^{-1} - 4e^{-1}}{9} = -\frac{7}{9e}$

c) slope of curve at this pt is also $-\frac{7}{9e}$

$$\textcircled{2} f(x) = \frac{x^2}{1+x^4}$$

$$\int_0^1 \left(\frac{df}{dx} \right) dx = f(x) \Big|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

$$\textcircled{3} \text{ let } y = \ln(1+x^2) \quad \text{b) } \int_1^2 \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) \Big|_1^2 = \frac{1}{2} (\ln 5 - \ln 2) = \ln \sqrt{\frac{5}{2}}$$

c) $y' = \frac{2x}{1+x^2}$

$$\textcircled{4} \text{ skip.}$$

$$\textcircled{5} \text{ a) } y = \frac{1}{\sqrt{x}}$$

$$\int \frac{1}{\sqrt{x}} dx = 2x^{\frac{1}{2}} + C, \quad y' = \frac{-1}{2x^{3/2}}$$

$$\text{b) } y = e^{-3x}$$

$$\int y = \frac{-e^{-3x}}{3} \quad y' = -3e^{-3x}$$

$$\text{c) } y = \frac{5}{x}$$

$$\int y = 5 \ln x \quad y' = -\frac{5}{x^2}$$

$$6 a) \left(\int \tan x^3 dx \right)' = \tan x^3$$

$$b) \left(\int_0^x \tan t^3 dt \right)' = \tan x^3 - \tan 0^3 = \tan x^3$$

c) 2nd FTC, and corollary to FTC, (pg 291)

$$7 a) \left(\ln(1 + e^{-3x}) \right)' = \frac{-3e^{-3x}}{1 + e^{-3x}}$$

$$b) (5 \sin x)' = 5 \cos x$$

$$c) (\sin 5x)' = 5 \cos 5x$$

8 skip.

$$9 a) \text{ cube root of } x \quad b) 10^x \quad c) x^2$$

10 skip

11 skip

Discuss
Necessity
of Problem 12
Diff of
Answers.

$$12 a) \int \frac{3}{x^2} dx = -\frac{3}{x} + C$$

$$b) \int \frac{3}{1+x^2} dx = 3 \tan^{-1} x + C$$

$$c) \int \frac{3x}{1+x^2} dx = \frac{3}{2} \ln(1+x^2) + C$$

13 } routine
14 }

$$21 \text{ Given } \lim_{h \rightarrow 0} \frac{\sinh h}{h} = 1 \text{ w.t.p. } \lim_{h \rightarrow 0} \frac{(1 - \cosh h)}{h} = 0$$

$$\lim_{h \rightarrow 0} \frac{(1 - \cosh h)(1 + \cosh h)}{h(1 + \cosh h)} = \lim_{h \rightarrow 0} \frac{(\sinh^2 h)}{h(1 + \cosh h)} = \lim_{h \rightarrow 0} \frac{\sinh h}{h} \cdot \frac{\sinh h}{1 + \cosh h}$$

$\downarrow$
 $1 \cdot 0 = 0$

(22) $y = \int e^{-x^3} dx$ find y'''

$$y' = e^{-x^3}, \quad y'' = -3x^2 e^{-x^3},$$

$$-6xe^{-x^3} + 9x^4 e^{-x^3} = y'''$$

(23) given $(\ln x)' = \frac{1}{x}$ wTP $(e^x)' = e^x$

a) let $y = e^x$ then $\ln y = x$

$$\frac{1}{y} \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = y = e^x$$

b) given $(\sin x)' = \cos x$ wTP $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$

let $y = \sin^{-1} x$, $\sin y = x$

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y} \quad \text{but} \quad \sin^2 y + \cos^2 y = 1 \quad \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}}$$

so $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

and $\sin y = x$ so

$$\cos y = \sqrt{1-x^2}$$

c) $(\tan x)' = \sec^2 x$ wTP $(\tan^{-1} x)' = \frac{1}{x^2+1}$

Same general method as b above.

(24) Since we move on the x axis $f(t)$ is

$$f(t) = e^{-t} \quad \text{velocity is 1st der.}$$

$$a) \left. f'(t) \right|_{t=2} = -e^{-t} \Big|_{t=2} = -e^{-2} \quad (\text{speed is } |vel|)$$

$$b) \text{ acceleration is 2nd deriv. } \left. f''(t) \right|_{t=2} = e^{-t} \Big|_{t=2} = e^{-2}$$

(25) $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e$

a) $n \rightarrow \infty$

b) 2.71828

(26) Skip

(27) $\int_{-1}^1 f(x) dx = F(x) \Big|_{-1}^1 = F(1) - F(-1) = F(1) - F(1) = 0$

3 students:
let $f(x) = x^{2m+1}$
 $\int f(x) dx = \frac{x^{2m+2}}{2m+2}$

Since $f(x)$ is odd then F is even fctn and even fctns have the property $f(x) = f(-x)$.

b) $\int_{-1}^1 x^3 e^{x^2} dx = 0$ Since $f(x)$ is odd if $f(-x) = -x f(x)$ then use a sub.

(28) a) $(5^{3x})'$

let $5 = e^{\ln 5}$

$y = 5^{3x} = e^{3x \ln 5}$

$y' = 3 \ln 5 \cdot e^{3x \ln 5}$

$y' = 3 \ln 5 \cdot 5^{3x}$

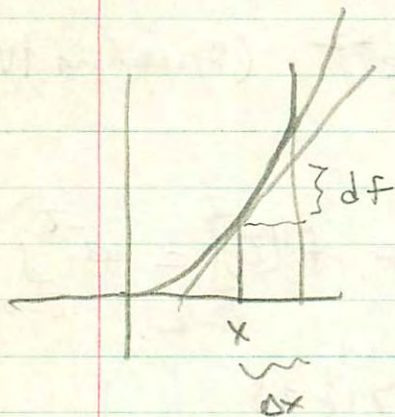
b) $\log_{10}(1+x^2) = \frac{\ln(1+x^2)}{\ln 10}$

$\frac{(1+x^2) \ln 10}{2x}$

Pg 305
30)

Possibly
they are
referring to $f'(x)$ as one
variable
and dx as the
other?

$$df = f'(x) dx$$



$$\textcircled{29} \quad \frac{d^2(x^5)}{dx^2} =$$

$$5x^4 \quad 20x^3 = y''$$

$$\text{b) } \ln x^2 = y \quad \text{c) } \frac{d^3\left(\frac{x}{1+x}\right)}{dx}$$

$$2 \ln x = y$$

$$y' = \frac{2}{x}$$

$$y'' = -\frac{2}{x^2}$$

$$y = \frac{x}{1+x}$$

$$y' = \frac{1+x-x}{(1+x)^2} = \frac{1}{(1+x)^2}$$

$$y'' = \frac{-2(1+x)}{(1+x)^3} = \frac{-2}{(1+x)^2}$$

$$\textcircled{31} \quad f = 5x + x^3 \quad x > 2 \quad \ln x = 0.1$$

$$f' = 5 + 3x^2$$

$$df = f'(x) \Delta x$$

$$f'(x) \Delta x$$

$$(5 + 3 \cdot 4)(0.1) = 1.7$$

$$\textcircled{32} \quad f = x^2 - 6x + 11$$

$$f' = 2x - 6 \stackrel{\text{set}}{=} 0 \quad x = 3$$

$$9 - 18 + 11$$

$$f'' = 2 \Big|_{x=3} \geq 0 \quad \text{min.}$$

2 is min value

$$\textcircled{33} \quad f = x^3 - 6x$$

$$f' = 3x^2 - 6 \stackrel{\text{set}}{=} 0 \quad x = \pm\sqrt{2}$$

$$f'' = 6x \Big|_{\pm\sqrt{2}} \quad f'' > 0 \quad x = +\sqrt{2} \quad \therefore \text{min at } x = \sqrt{2}$$

$$2\sqrt{2} - 6\sqrt{2} = -4\sqrt{2} \quad \text{min value}$$

$$\textcircled{34}$$

$$3 = 5e^{.035}$$

$$\ln 3 = \ln 5 + .035 \ln e$$

$$\ln 3 - \ln 5 = .035$$

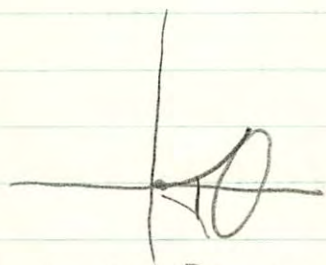
$$\frac{100}{3} \ln \frac{3}{5} = t \quad \frac{3}{100}$$

35) these undo each other.

36) a) $\int_0^3 4x^3 dx = x^4 \Big|_0^3 = 81 - 0 = 81 \text{ grams.}$

b) same as a.

c) $y = x^{3/2}$
 $y' = \frac{3}{2} x^{1/2}$
 $y'' = \frac{3}{4} x^{-1/2}$ concave up



cross section is circle area is πr^2 and
 $r = x^{3/2}$ so

$$\pi \int_0^3 x^3 dx = \pi \left(\frac{x^4}{4} \Big|_0^3 \right) = \frac{81\pi}{4}$$

37) Since $f(t)$ is rate of depure, $\int f(t) dt$ is total depreciation.

a) $g(t) = \frac{A + \int_0^t f(t) dt}{t}$

$$\frac{\text{Repair costs} + \text{loss due to depure}}{\text{Time}} = \text{Cost/unit of Time.}$$

You want to minimize cost so you would want to minimize $g(t)$

b) $g'(t) = \frac{t f(t) - (A + \int_0^t f(t) dt)}{t^2} \stackrel{\text{set}}{=} 0$

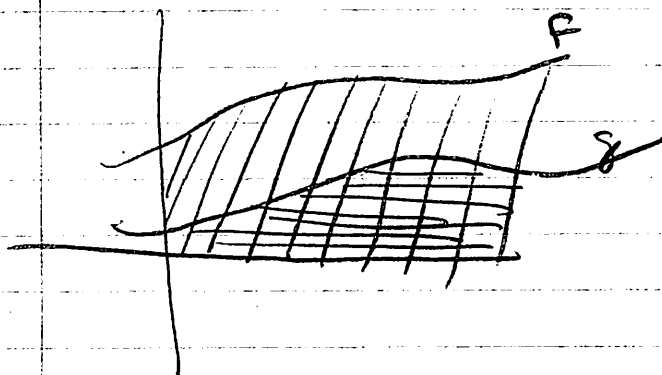
c) $t f(t) = A + \int_0^t f(t) dt$

$f(t) = \frac{A + \int_0^t f(t) dt}{t} = g(t)$ by (a) abv.

d) Rate of depure = current cost $\uparrow$ Yes.
 if $f(t) > g(t)$ then too much depure has taken place.
 if $f(t) < g(t)$ then full use of machinery was not taken.

38) $\int f(x) dx$; $\int g(x) dx$; $\int [f(x) - g(x)] dx$

a) Area



$\int f$ all area under curve

$\int g$ area under g .

$\int (f - g)$ area between f and g

$$\int (f(x) - g(x)) \quad f(x) - g(x) = h(x)$$

So we still end up w/ area between.

So $\int [f(x) - g(x)] dx = \int f(x) dx - \int g(x) dx$.

b) $\int (f(x) g(x)) dx$, Product.

I can't determine what area + area would be, whatever it is probably is not the area under the product of the two fct's.

39) a) $\int_0^b e^{-x} dx = -e^{-x} \Big|_0^b = -e^{-b} + 1$

b) $\lim_{b \rightarrow \infty} (-e^{-b} + 1) = 1$. The area approaches 1 sq unit.

$$\frac{e^{-b} - 1}{e^b}$$

$$1 - \frac{1}{e^b}$$

Typo

305

(40)

$$\int_0^{\sqrt{\pi}} 2x \sin x^2 dx$$

or

$$\int_0^{\sqrt{\pi}} 2 \sin x^2 dx$$

let $u = -\cos x^2$

then $du = 2x \sin x^2 dx$

↑ Not this one, it needs the factor of x to evaluate

$$\int_0^{\sqrt{\pi}} (-\cos x^2)' dx = -\cos x^2 \Big|_0^{\sqrt{\pi}} = -(-1) + 1 = 2$$

too hard.

(41) $y = e^{-x} \sin x$

$$y = x e^{-x} \frac{\sin x}{x}$$

$\lim_{x \rightarrow 0} = 0$

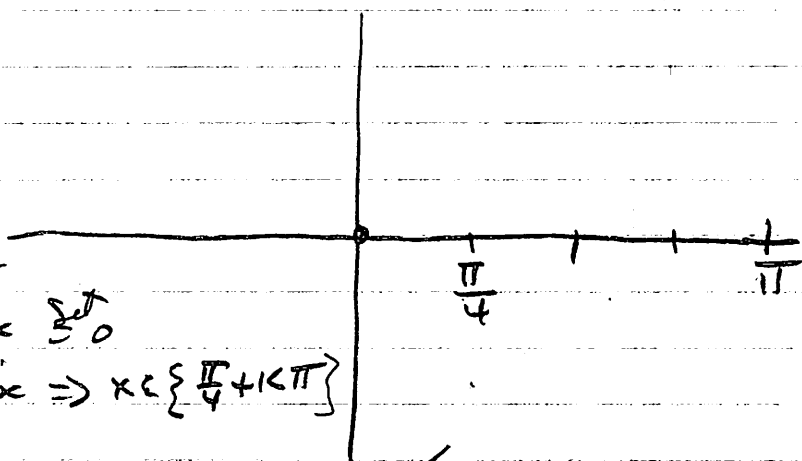
$$y' = -e^{-x} \sin x + e^{-x} \cos x \stackrel{?}{=} 0$$

$$\sin x = \cos x \Rightarrow x \in \left\{ \frac{\pi}{4} + k\pi \right\}$$

$$y'' = e^{-x} \sin x - e^{-x} \cos x - e^{-x} \cos x + e^{-x} \sin x$$

$$y'' = -2e^{-x} \cos x \Rightarrow \min x \in \left\{ \frac{5\pi}{4} + 2k\pi \right\}$$

$$\max x \in \left\{ \frac{\pi}{4} + 2k\pi \right\}$$



420

$$\lim_{x \rightarrow \infty} \left(\frac{2^x + 8^x}{2} \right)^{\frac{1}{x}}$$

$$\ln z = \frac{\ln(2^x + 8^x) - \ln 2}{x}$$

$$\frac{0}{0}$$

42C

$$e^x \approx 1+x$$

$$\lim_{x \rightarrow 0} \frac{e^x}{1+x} = 1$$

so e^x is approximated by $1+x$

on T182
2nd table set

$$T_{blmin} = .035$$

$$\Delta T_{BL} = -.01$$

2nd table

to see values
close in on zero

$$2^x = e^{\ln 2^x}$$

$$= e^{x \ln 2} \approx 1 + x \ln 2$$

$$8^x = 2^{3x} = e^{\ln 2^{3x}}$$

$$= e^{3x \ln 2} \approx 1 + 3x \ln 2$$

$$\frac{2^x + 8^x}{2} \approx$$

$$\frac{1 + x \ln 2 + 1 + 3x \ln 2}{2}$$

$$\frac{2 + 4x \ln 2}{2}$$

$$1 + 2x \ln 2$$

$$\lim_{x \rightarrow 0} \left(\frac{2^x + 8^x}{2} \right)^{\frac{1}{x}} \approx$$

$$\lim_{x \rightarrow 0} (1 + 2x \ln 2)^{\frac{1}{x}}$$

$$\lim_{x \rightarrow 0} \left[(1 + 2x \ln 2)^{\frac{1}{2x \ln 2}} \right]^{2 \ln 2}$$

$$e^{2 \ln 2}$$

$$e^{\ln 4}$$

4. QED.

$$(42) \quad a) \lim_{h \rightarrow 0} \frac{(1+h)^3 - (1+h)^2}{2+h} = \frac{1-1}{2} = 0.$$

$$b) \lim_{x \rightarrow \infty} \frac{2x^3 + e^{-x}}{x^3 - 1} = \lim_{x \rightarrow \infty} \frac{2 + \frac{1}{x^3}e^{-x}}{1 - \frac{1}{x^3}} \Rightarrow 2$$

$$c) \lim_{x \rightarrow 0} \left(\frac{2^x + 8^x}{2} \right)^{\frac{1}{x}} = \lim_{x \rightarrow 0} \left(\frac{2^x + 2^{3x}}{2} \right)^{\frac{1}{x}} = \lim_{x \rightarrow 0} \left[\frac{2^x (1 + 2^{2x})}{2} \right]^{\frac{1}{x}}$$

$$\lim_{x \rightarrow 0} \left[2^x \cdot \frac{(1 + 2^{2x})}{2} \right]^{\frac{1}{x}} = \lim_{x \rightarrow 0} 2 \cdot \lim_{x \rightarrow 0} \left[\frac{1 + 2^{2x}}{2} \right]^{\frac{1}{x}} =$$

$$2 \cdot \lim_{x \rightarrow 0} \frac{(1 + 2^{2x})^{\frac{1}{x}}}{2^{\frac{1}{x}}} = \lim_{x \rightarrow 0} \frac{(1 + 4^x)^{\frac{1}{x}}}{2^{\frac{1}{x}}}$$

$$\lim_{x \rightarrow 0} \left[\frac{4^x}{2} \right]^{\frac{1}{x}} < \lim_{x \rightarrow 0} \frac{(1 + 4^x)^{\frac{1}{x}}}{2^{\frac{1}{x}}} < \lim_{x \rightarrow 0} \left(\frac{4^x + 4^x}{2} \right)^{\frac{1}{x}}$$

$$\lim_{x \rightarrow 0} \left[\frac{2^{2x}}{2} \right]^{\frac{1}{x}}$$

$$\lim_{x \rightarrow 0} \left[\frac{2 \cdot 4^x}{2} \right]^{\frac{1}{x}}$$

$$\lim_{x \rightarrow 0} \left[2^{2x-1} \right]^{\frac{1}{x}}$$

$$\lim_{x \rightarrow 0} 4$$

$$\lim_{x \rightarrow 0} 2^{2 - \frac{1}{x}}$$

$$< 4$$

$$\leq \left[\frac{2^x + 2^{3x}}{2} \right]^{\frac{1}{x}} \leq \left[\frac{2^{3x} + 2^{3x}}{2} \right]^{\frac{1}{x}}$$

$$\leq \left[\frac{2 \cdot 2^{3x}}{2} \right]^{\frac{1}{x}}$$

$$\leq 2^3$$

$$\leq 8$$

$$8 \leq$$

42 d. $\lim_{t \rightarrow 0^+} \frac{\int_0^t \sqrt{\sin 3x} \, dx}{\int_0^t \sqrt{x} \, dx} \quad \frac{0}{0} \text{ so use l'Hopital's Rule, take deriv.}$

Now even, using 2nd FTC $\Rightarrow \frac{d(\int_a^x f(t) \, dt)}{dx} = f(x)$.

$$\lim_{t \rightarrow 0^+} \frac{\sqrt{\sin 3t}}{\sqrt{t}} = \lim_{t \rightarrow 0} \frac{\sqrt{3} \sqrt{\sin 3t}}{\sqrt{3} \sqrt{t}} =$$

$$\lim_{t \rightarrow 0} \sqrt{3} \sqrt{\frac{\sin 3t}{3t}} = \sqrt{3} \lim_{t \rightarrow 0} \sqrt{\frac{\sin 3t}{3t}} = \sqrt{3} \cdot 1 = \sqrt{3}$$

42 e) $\lim_{t \rightarrow \infty} \frac{\int_1^t \sqrt[3]{x^6 + x + 1} \, dx}{\int_3^t \sqrt{x^4 + x^3 + 5} \, dx} \quad \frac{\infty}{\infty} \text{ use l'Hopital's rule.}$

$$\lim_{t \rightarrow \infty} \frac{\sqrt[3]{t^6 + t + 1}}{\sqrt{t^4 + t^3 + 5}} = \lim_{t \rightarrow \infty} \frac{\sqrt{1 + \frac{1}{t^5} + \frac{1}{t^6}}}{\sqrt{1 + \frac{1}{t^3} + \frac{5}{t^3}}} = 1$$

42 f) $\lim_{t \rightarrow 2} \frac{\int_4^{t^2} e^{x^2} \, dx}{\int_8^{t^3} e^{x^3} \, dx} \quad \frac{0}{0} \text{ use l'Hopital's Rule,}$

$$\lim_{t \rightarrow 2} \frac{2t e^{t^4}}{3t^2 e^8} = \lim_{t \rightarrow 2} \frac{2t e^{t^4 - t^8}}{3t^2} = \lim_{t \rightarrow 2} \frac{2t e^{t^4(1-t^4)}}{3t^2} =$$

$$\frac{1}{3} e^{16(31)} = \frac{e^{-496}}{3}$$

$$(43) \int x^n \tan^{-1} ax \, dx =$$

$$\frac{x^{n+1}}{n+1} \tan^{-1} ax - \frac{a}{n+1} \int \frac{x^{n+1} dx}{1+a^2 x^2}$$

$$x^n \tan^{-1} ax + \frac{x^{n+1}}{n+1} \cdot \frac{a}{1+a^2 x^2} - \frac{a}{n+1} \left[\frac{x^{n+1}}{1+a^2 x^2} \right]$$

$$x^n \tan^{-1} ax$$

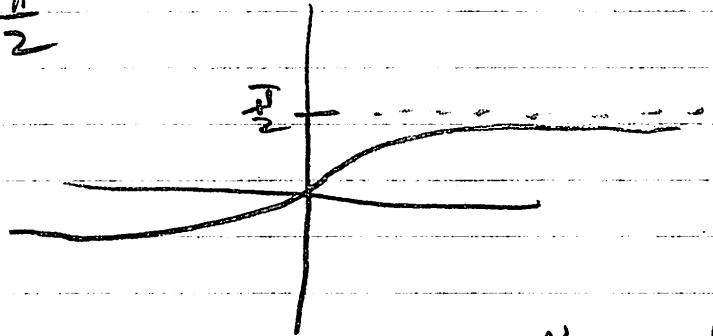
$$48 \quad A(t) = \int_0^t \frac{1}{1+x^2} dx \quad \lim_{t \rightarrow \infty} A(t)$$

$$A(t) = \tan^{-1} t - \tan^{-1} 0 \quad \tan^{-1} 0 = x$$

$$\tan^{-1} t = 0 \quad \tan x = 0$$

$$x = 0.$$

$$\lim_{t \rightarrow \infty} \tan^{-1} t = \frac{\pi}{2}$$



$$? \quad 49 \quad a_n = \frac{n}{(n+1)^2} + \frac{n}{(n+2)^2} + \dots + \frac{n}{(n+n)^2} \quad \frac{n}{4n^2} = \frac{1}{4n}$$

$$a_1 = \frac{1}{(1+1)^2} = \frac{1}{4} = .25 \quad \frac{2}{4^2}$$

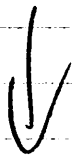
$$a_2 = \frac{2}{3^2} + \frac{2}{4^2} = \frac{2}{9} + \frac{1}{8} = .347$$

$$a_3 = \frac{3}{4^2} + \frac{3}{5^2} + \frac{3}{6^2} = \frac{3}{16} + \frac{3}{25} + \frac{1}{12} = .391$$

- (50) If we knew the cross section we would integrate to determine the volume.
Since we know the volume $(V(x))$ we must take the derivative.

(51) let investment $= f(t)$

$$\frac{d f(t)}{dt} = k f(t).$$



$A = \text{investment capital}$ $f(t) = \frac{dA}{dt} \cdot \text{rate of investment}$

(51)

WTS $f(t) = k \left(A + \underbrace{\int_0^t f(x) dx}_{\uparrow} \right) \quad t \geq 0$

$\left(\frac{du}{u} = du \right)$

This is Addition return from investment which will be reinvested.

$f'(t) = k f(t)$

$\int \frac{df(t)}{f} = \int k dt$

$\ln f = kT$
 $f = e^{kT}$

(52) $v = \frac{1}{1+x^2} \quad v' = \frac{-2x}{(1+x^2)^2}$

$v'' = \frac{-2(1+x^2)^2 + (2x^2 \cdot 2(1+x^2))}{(1+x^2)^4}$

$v' \Big|_{\frac{\sqrt{3}}{3}} = \frac{3\sqrt{3}}{8}$

$= \frac{(1+x^2) \cdot (-2(1+x^2) + 8x^2)}{(1+x^2)^4}$

$= \frac{-2 + 6x^2}{(1+x^2)^3} \stackrel{=0}{=} x = \pm \frac{\sqrt{3}}{3}$

$x < -\frac{\sqrt{3}}{3} \quad \frac{-2(1-3x^2)}{+} = +$

$x < +\frac{\sqrt{3}}{3}$ Same

max.

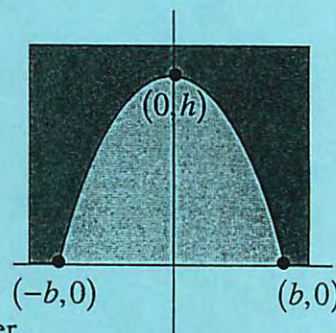
$x > -\frac{\sqrt{3}}{3} \quad \frac{-1}{+} = -$

$v' \Big|_{x=\pm \frac{1}{\sqrt{3}}} = \frac{\pm \frac{2}{\sqrt{3}}}{(1+\frac{1}{3})^2} = \pm \frac{2}{\frac{16}{9}} = \pm \frac{3}{8}$

This "honor quiz" is to be done by you totally on your own.

You are not allowed to discuss your solution with anyone until the quiz has been turned in Monday morning, February 7, not later than 7:55 in the morning. The value of this quiz is 3 homework assignments. Answer the questions on this paper. You may use the back if necessary.

An arched window with base width $2b$ and height h is to be set into a wall. The arch is to be either an arc of a parabola or a half-cycle of a cosine curve.



- If the arch is an arc of a parabola, write an equation for the parabola relative to the coordinate system shown in the figure. Explain.
- If the arch is a half-cycle of a cosine curve, write an equation for the cosine curve relative to the coordinate system shown in the figure. Explain.
- Of these two window designs, which has the greater area? Justify your answer.

a) $y = -Ax^2 + h$ when $x=0, y=h$
 Set $y=0$ let $A > 0$. Since graph opens downwards
 $-Ax^2 + h = 0$
 $x^2 = \frac{h}{A}$
 $x = \pm \sqrt{\frac{h}{A}} = b$
 $\frac{h}{A} = b^2$
 $A = \frac{h}{b^2}$

b) Period = $4b$ $h = \text{amplitude}$
 $\frac{2\pi}{B} = 4b$
 $B = \frac{\pi}{2b}$ Amplitude is h
 $y = h \cos \frac{\pi}{2b} x$

$\therefore y = -\frac{h}{b^2} x^2 + h$

c) $2 \int_0^b -\frac{h}{b^2} x^2 + h dx$
 $2 \left(-\frac{h}{b^2} \frac{x^3}{3} + hx \right) \Big|_0^b$
 $2 \left(-\frac{h}{b^2} \frac{b^3}{3} + hb \right)$
 $2 \left(-\frac{hb}{3} + hb \right)$
 $\frac{4hb}{3}$

$2 \int_0^h h \cos \frac{\pi}{2b} x dx$
 $2 h \frac{2b}{\pi} \sin \frac{\pi}{2b} x \Big|_0^h$
 $\frac{4hb}{\pi}$

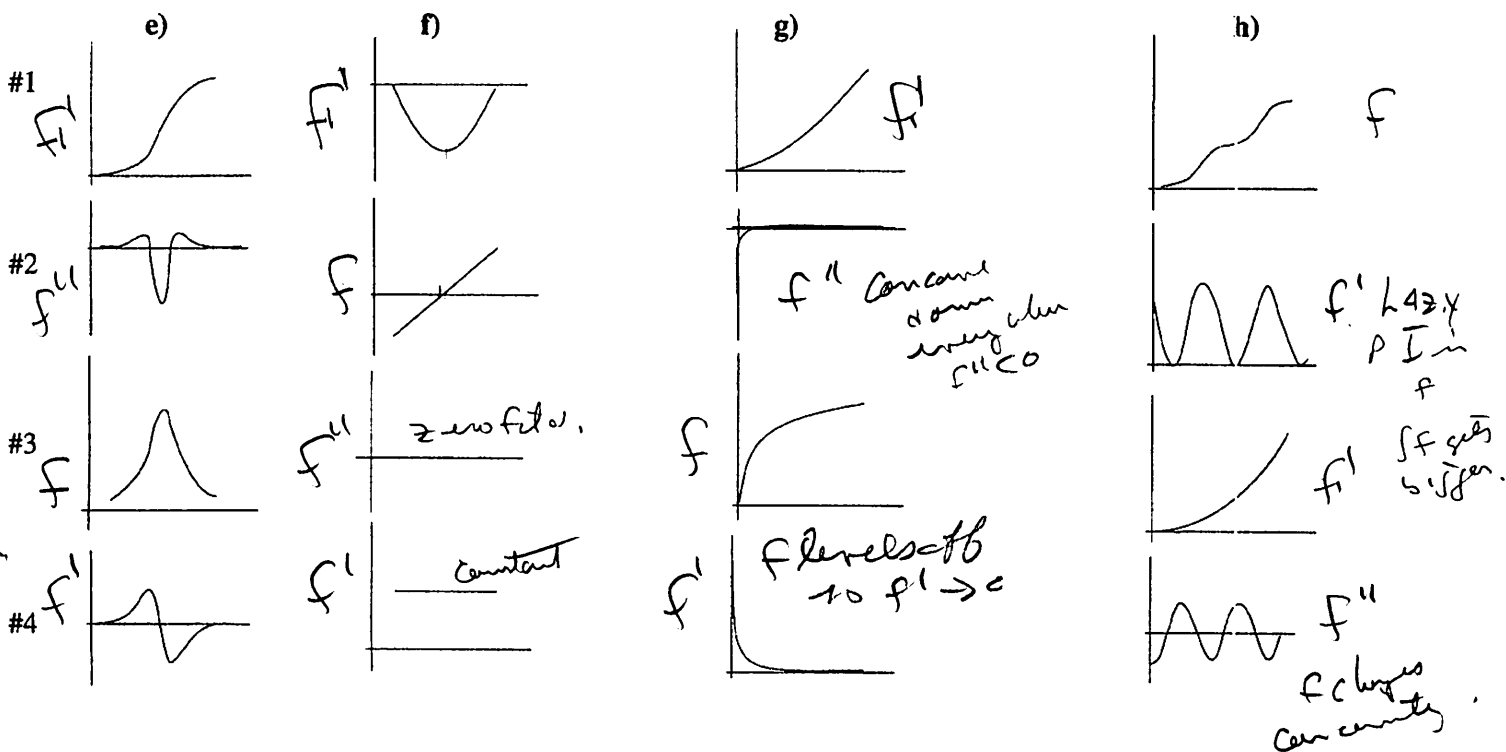
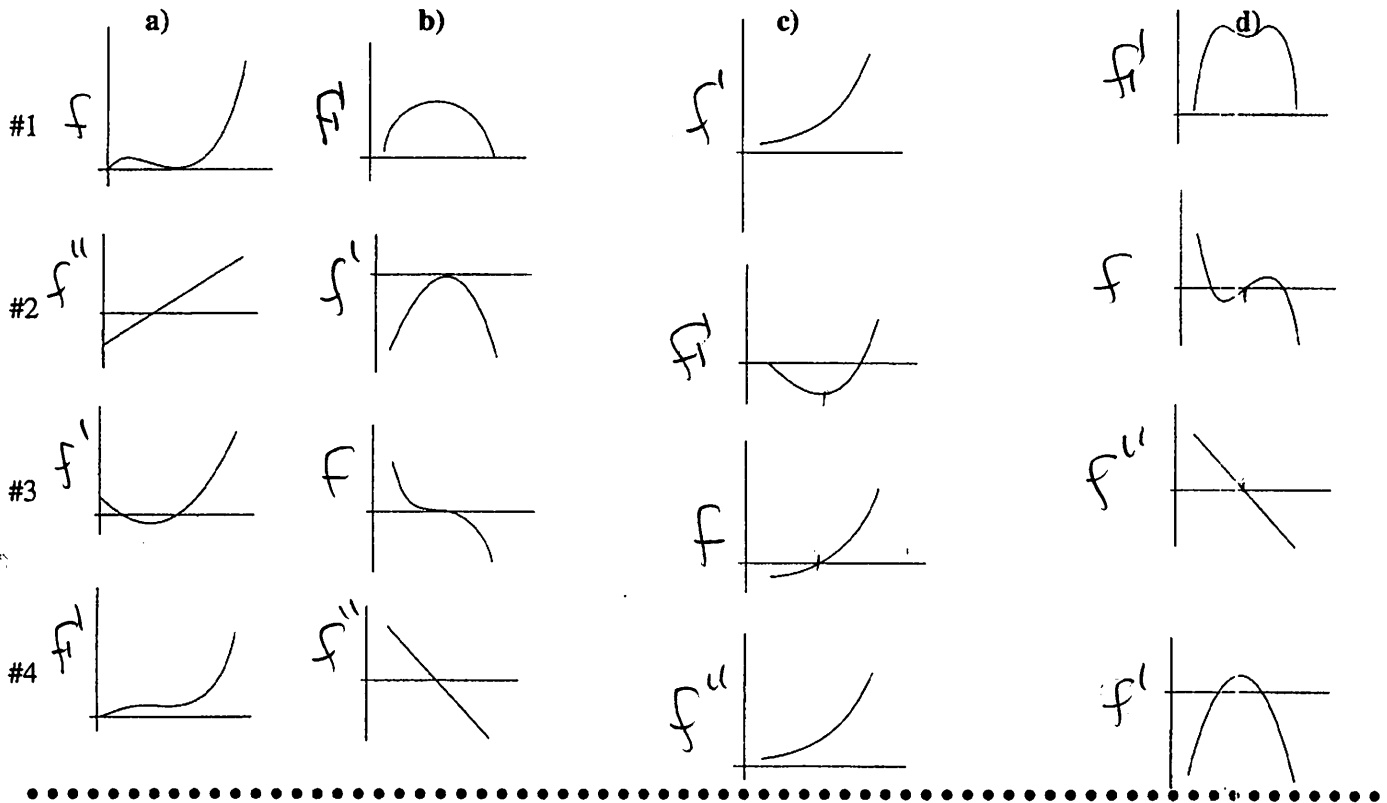
The area under the parabola is more because $3 < \pi$ so we are dividing by a smaller number.

Calculus

$F f f' f''$ (un)ctions

J Mumaugh

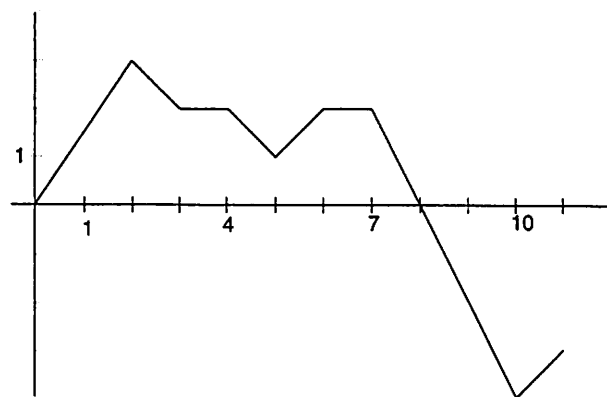
Each of the following columns is contains four related functions in random order, the graphs of $f(x)$, $f'(x)$, $f''(x)$, and the function $F(x)$ (whose derivative is $f(x)$ ie. $F'(x) = f(x)$). Label each graph with the correct function.



Translating Graphenese to English

I

A bug is crawling along a straight wire. The velocity, $v(t)$, of the bug at time t , $0 \leq t \leq 11$, is given in the graph on the right.

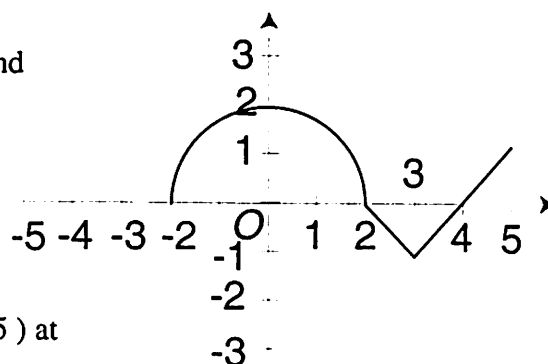


- According to the graph, at what time t does the bug change direction?
- According to the graph, at what time t is the speed of the bug greatest?
- How far did the bug travel?
- How far is the bug from its starting point?
- Why are the answers to parts "c" and "d" different?
- Compute: $\int_0^8 v(t) dt$, $\int_8^{11} v(t) dt$, and $\int_0^{11} v(t) dt$

II

The graph of a function f consists of a semicircle and two line segments as shown. Let g be the function

given by $g(x) = \int_0^x f(t) dt$.

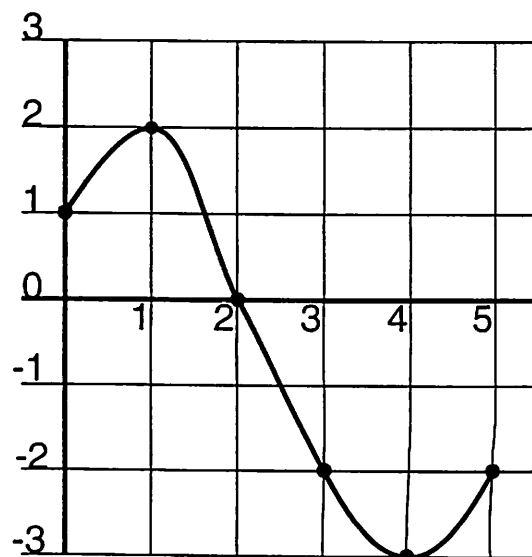


- Find $g(3)$.
- Find all values of x on the open interval $(-2, 5)$ at which g has a relative maximum. Justify your answer.
- Write an equation for the line tangent to the graph of g at $x = 3$.
- Find the x -coordinate of each point of inflection of the graph of g on the open interval $(-2, 5)$. Justify your answer.

III

Let f be a function whose domain is the closed interval $[0, 5]$. The graph of f is shown on the right.

Let $h(x) = \int_0^{\frac{x+3}{2}} f(t) dt$



- Find the domain of h .
- Find $h'(2)$.
- At what x is $h(x)$ a minimum? Show the analysis that leads to your conclusion.

$$g(x) = \int_{-1}^{2x} f(\tau) d\tau$$

$$g(-\frac{1}{2}) = \int_{-1}^{-1} f(\tau) d\tau = 0$$

$$g(0) = \int_{-1}^0 f(\tau) d\tau = -\frac{1}{2}$$

Below
x axis

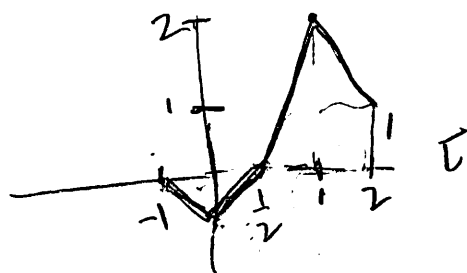
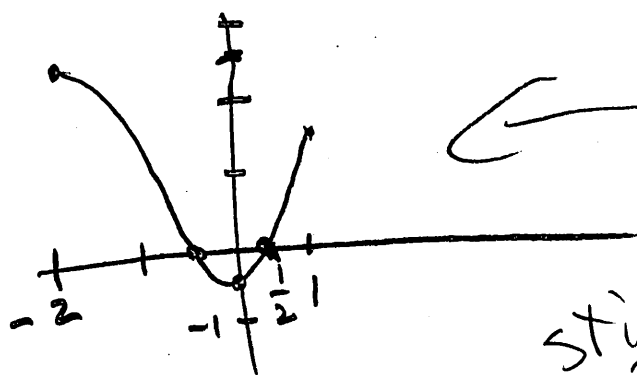
$$g(\frac{1}{2}) = \int_{-1}^1 f(\tau) d\tau = -\frac{1}{2} + \frac{1}{2} = 0$$

$$g(1) = \int_{-1}^2 f(\tau) d\tau = 1\frac{1}{2}$$

$$-2 \leq x \leq 1$$

Carl E. Hunt
553 -
2846

There
is
not
a
picture
with
it.



Still x
Not quite
done
Troubles.

$$-1 \leq \tau \leq 2$$

$$-\frac{1}{2} \leq x \leq 1$$

Should
rise

$$2x = \frac{1}{2}$$

Give pict on next page.

25pts

Let f be a fctn whose domain is the closed interval $[0, 5]$. The graph of f is given.

$$\text{Let } h(x) = \int_0^{x+1} f(t) dt$$

~~Calc ch to~~
~~Calc ch 78~~
~~7777~~
worksheet

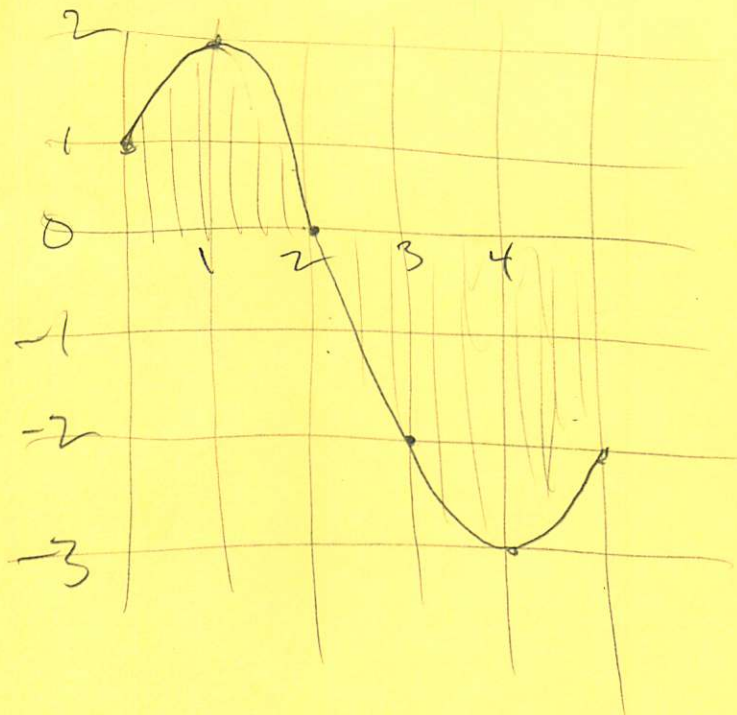
A) Explain how you know the Domain of h . (D_h)
is given by $-6 \leq x \leq 4$. 5pts

B) Find $h'(2)$

5pts

C) At what x value does $h(x)$ have its
absolute minimum?

15pts



$$x \in [0, 5]$$

$$h(x) = \int_0^{\frac{x}{2}+3} f(\tau) d\tau$$

A) Domain_h = ? $0 \leq \frac{x}{2} + 3 \leq 5$

D_h $x \in -6 \leq x \leq 4$ $-3 \leq \frac{x}{2} \leq 2$
 $-6 \leq x \leq 4$

B) $h'(2) = ?$

$$h'(x) = f\left(\frac{x}{2} + 3\right) \cdot \frac{1}{2}$$

$x = 2$

← drop inside.

$$\frac{1}{2} f(1+3)$$

$$\frac{1}{2} f(4)$$

$$\frac{1}{2} (-3) = -\frac{3}{2}$$

from pict.

d) h has min?

$$h(x) = \int_0^{x+3} f(t) dt$$

$$h' = \frac{1}{2} f\left(\frac{x+3}{2}\right) \stackrel{\text{set}}{=} 0$$

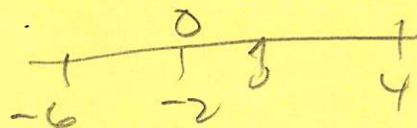
$$f\left(\frac{x+3}{2}\right) = 0$$

$$\frac{x+3}{2} \stackrel{\text{set}}{=} 2$$

$$\frac{x}{2} = -1$$

$$x = -2$$

← from picture when $f=0$



$$h'' = \frac{1}{2} f'\left(\frac{x+3}{2}\right) \text{ is slope of } f \text{ at}$$

is negative. The slope of the f at x is decreasing so h'' is < 0 so $x = -2$ is a MAX f'' test.

$\therefore$ min must occur at END PTS.

$x = -6$ min END PT inc to it

$x = 4$ min END PT Dec to it.

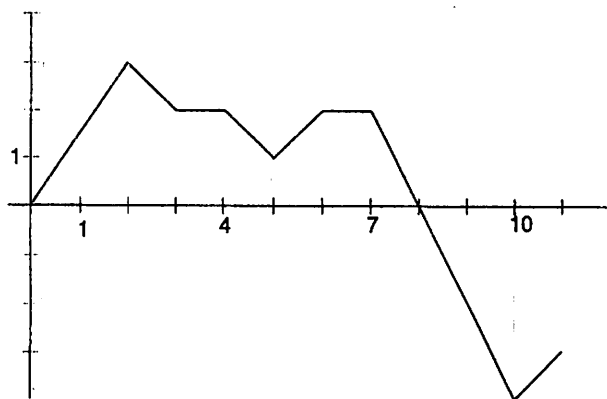
Since area below the axis is greater than the area above the axis, the absolute min occurs at $x = 4$.

Translating Graphenese to English



I

A bug is crawling along a straight wire. The velocity, $v(t)$, of the bug at time t , $0 \leq t \leq 11$, is given in the graph on the right.

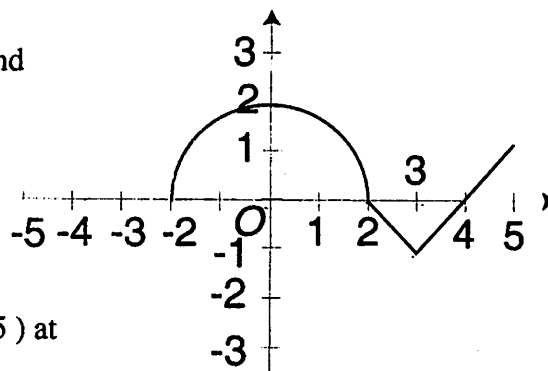


- According to the graph, at what time t does the bug change direction?
- According to the graph, at what time t is the speed of the bug greatest?
- How far did the bug travel?
- How far is the bug from its starting point?
- Why are the answers to parts "c" and "d" different?
- Compute: $\int_0^8 v(t) dt$, $\int_8^{11} v(t) dt$, and $\int_0^{11} v(t) dt$

II

The graph of a function f consists of a semicircle and two line segments as shown. Let g be the function

given by $g(x) = \int_0^x f(t) dt$.



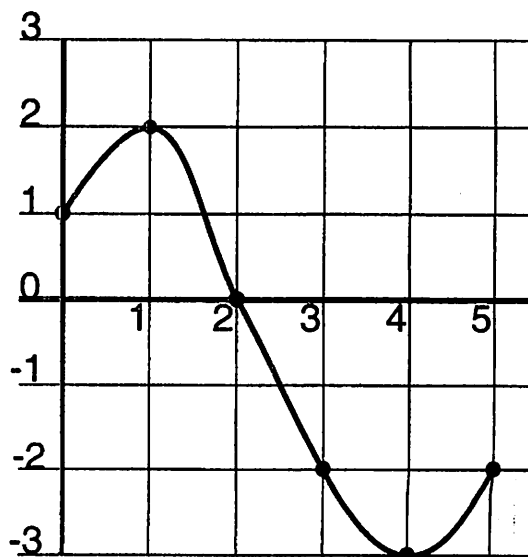
- Find $g(3)$.
- Find all values of x on the open interval $(-2, 5)$ at which g has a relative maximum. Justify your answer.
- Write an equation for the line tangent to the graph of g at $x = 3$.
- Find the x -coordinate of each point of inflection of the graph of g on the open interval $(-2, 5)$. Justify your answer.

III

Let f be a function whose domain is the closed interval $[0, 5]$. The graph of f is shown on the right.

Let $h(x) = \int_0^{\frac{x}{2}+3} f(t) dt$

- Find the domain of h .
- Find $h'(2)$.
- At what x is $h(x)$ a minimum? Show the analysis that leads to your conclusion.



Each of the following columns is contains four related functions in random order, the graphs of $f(x)$, $f'(x)$, $f''(x)$ and the function $F(x)$ (whose derivative is $f(x)$ ie. $F'(x) = f(x)$) . Label each graph with the correct function.

